

# EXISTENCE AND A PRIORI BOUNDS FOR ELECTROSTATIC KLEIN-GORDON-MAXWELL SYSTEMS IN FULLY INHOMOGENEOUS SPACES

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**ABSTRACT.** We prove existence and uniform bounds for electrostatic Klein-Gordon-Maxwell systems in the inhomogeneous context of a compact Riemannian manifold when the mass potential, balanced by the phase, is small in a quantified sense. Phase compensation for electrostatic Klein-Gordon-Maxwell systems and the positive mass theorem are used in a crucial way.

Electrostatic Klein-Gordon-Maxwell systems are electrostatic derivations of the Klein-Gordon-Maxwell systems which, in turn, are a special case of the Yang-Mills-Higgs equation. They arise naturally in quantum mechanics. Roughly speaking, Klein-Gordon-Maxwell systems provide a dualistic model for the description of the interaction between a charged relativistic particle of matter and the electromagnetic field that it generates. The electromagnetic field is both generated by and drives the particle field. In the electrostatic form of the Klein-Gordon-Maxwell systems, writing the matter particle as a standing wave  $ue^{i\omega t}$ , it is characterized by the property that  $u$  solves the electrostatic Klein-Gordon-Maxwell systems we investigate in this paper with a gauge potential  $v$ . In what follows we let  $(M, g)$  be a smooth compact 3-dimensional Riemannian manifold and  $a > 0$  be a smooth positive function in  $M$ . Let  $\omega_a$  be given by

$$\omega_a = \sqrt{\min_M a} . \quad (0.1)$$

Given real numbers  $q > 0$ ,  $\omega \in (-\omega_a, \omega_a)$ ,  $\lambda \geq 0$ , and  $p \in (2, 6]$ , the electrostatic Klein-Gordon-Maxwell systems we investigate in this paper are written as

$$\begin{cases} \Delta_g u + au = u^{p-1} + \omega^2 (qv - 1)^2 u \\ \Delta_g v + (\lambda + q^2 u^2) v = qu^2 , \end{cases} \quad (0.2)$$

where  $\Delta_g = -\operatorname{div}_g \nabla$  is the Laplace-Beltrami operator. The system (0.2) is energy critical when  $p = 6$ , and subcritical when  $p \in (2, 6)$ . In the classical physical setting,  $a = m_0^2$ ,  $m_0$  is the mass of the particle, the coercivity constant  $\lambda = 0$ ,  $q$  is the charge of the particle,  $u$  is the field associated to the particle,  $\omega$  is the temporal frequency (referred to as the phase in the sequel), and  $v$  is the electric potential. The constant  $\lambda$  in this paper can be interpreted in terms of the Maxwell-Proca theory (see Hebey and Truong [42]). In what follows we let  $S_g$  stand for the scalar curvature of  $g$ . Also we let  $\mathcal{S}_p(\omega)$  be the set consisting of the positive smooth solutions  $\mathcal{U} = (u, v)$

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of (0.2) with phase  $\omega$  and nonlinear term  $u^{p-1}$ . Namely,

$$\mathcal{S}_p(\omega) = \left\{ (u, v) \text{ smooth s.t. } u > 0, v > 0, \text{ and } (u, v) \text{ solve (0.2)} \right\}. \quad (0.3)$$

Given  $\omega \in [0, \omega_a)$ , we let

$$K_0(\omega) = (-\omega_a, -\omega] \bigcup [\omega, \omega_a), \quad (0.4)$$

where  $\omega_a$  is as in (0.1). When  $\omega = 0$ ,  $K_0(0)$  is the interval  $K_\varepsilon(0) = (-\omega_a, \omega_a)$ . For  $\theta \in (0, 1)$ , and  $\mathcal{U} = (u, v)$ , we let

$$\|\mathcal{U}\|_{C^{2,\theta}} = \|u\|_{C^{2,\theta}} + \|v\|_{C^{2,\theta}}. \quad (0.5)$$

We recall that  $(M, g)$  is said to be conformally diffeomorphic to the unit 3-sphere  $(S^3, g_0)$  if there exists a diffeomorphism  $\varphi : S^3 \rightarrow M$  such that  $\varphi^*g = u^4g_0$  for some smooth positive function  $u$  in  $S^3$ . We prove below the existence of smooth positive solutions and the existence of uniform bounds for (0.2) in the subcritical cases  $p \in (2, 6)$  without any conditions, and in the critical case  $p = 6$  assuming that the mass potential, balanced by the phase, is smaller than the geometric threshold potential of the conformal Laplacian. Our main result is as follows. Closely related estimates are derived in Theorem 2.1 in Section 2.

**Theorem 0.1.** *Let  $(M, g)$  be a smooth compact 3-dimensional Riemannian manifold and  $a > 0$  be a smooth positive function in  $M$ . Let  $q > 0$ ,  $\omega \in (-\omega_a, \omega_a)$ ,  $\lambda \geq 0$ , and  $p \in (2, 6]$ , where  $\omega_a$  is as in (0.1). When  $p = 6$  assume*

$$a \leq k\lambda\omega^2 + \frac{1}{8}S_g \quad (0.6)$$

*in  $M$  for some  $k > 0$  such that  $k\lambda < 1$ . Then (0.2) possesses a smooth positive solution. Moreover, for any  $p \in (2, 6)$ , and any  $\theta \in (0, 1)$ , there exists  $C > 0$  such that for any  $\omega' \in K_0(0)$ , and any  $\mathcal{U} \in \mathcal{S}_p(\omega')$ ,  $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$ , where  $\mathcal{S}_p(\omega')$  is as in (0.3),  $K_0(0)$  is as in (0.4), and  $\|\cdot\|_{C^{2,\theta}}$  is as in (0.5). Assuming again (0.6), with the property that (0.6) is strict at least at one point if  $(M, g)$  is conformally diffeomorphic to the unit 3-sphere and  $\omega\lambda = 0$ , there also holds that for any  $\theta \in (0, 1)$ ,  $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$  for all  $\mathcal{U} \in \mathcal{S}_6(\omega')$  and all  $\omega' \in K_0(\omega)$ , where  $C > 0$  does not depend on  $\omega'$  and  $\mathcal{U}$ .*

There are several consequences to our theorem. The first obvious one is that solutions in the subcritical case exist for all phases and are uniformly bounded in  $C^{2,\theta}$ . For the sake of clearness we restate this result in the following corollary.

**Corollary 0.1** (Subcritical Case). *Let  $(M, g)$  be a smooth compact 3-dimensional Riemannian manifold and  $a > 0$  be a smooth positive function in  $M$ . Let  $q > 0$ ,  $\lambda \geq 0$ , and  $p \in (2, 6)$ . For any  $\omega \in (-\omega_a, \omega_a)$  there exists a smooth positive solution of (0.2). Moreover, for any  $\theta \in (0, 1)$ , there exists  $C > 0$  such that  $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$  for all  $\omega \in (-\omega_a, \omega_a)$ , and all  $\mathcal{U} \in \mathcal{S}_p(\omega)$ .*

Two notable consequences of the theorem concern the critical case where it holds that  $p = 6$ . Assuming  $\lambda > 0$ , the first consequence we discuss, which provides a perfect illustration of phase compensation, is that if the oscillation of  $a$ , given by  $\text{Osc}(a) = \max_M a - \min_M a$ , is not too large, then there always are solutions of our system for sufficiently large phases and such solutions are again uniformly bounded in  $C^{2,\theta}$ .

**Corollary 0.2** (Critical Case 1). *Let  $(M, g)$  be a smooth compact 3-dimensional Riemannian manifold and  $a > 0$  be a smooth positive function in  $M$ . Let  $q > 0$  and  $\lambda > 0$ . Suppose  $\text{Osc}(a) < \frac{1}{8} \min_M S_g$ . Then there exists  $\varepsilon > 0$  such that for any  $\omega \in K_0(\omega_a - \varepsilon)$ , (0.2) possesses a smooth positive solution when  $p = 6$ . Moreover, for any  $\theta \in (0, 1)$ , there exists  $C > 0$  such that  $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$  for all  $\omega \in K_0(\omega_a - \varepsilon)$ , and all  $\mathcal{U} \in \mathcal{S}_6(\omega)$ .*

The last consequence of the theorem we discuss, still dealing with the critical case where  $p = 6$ , concerns the more restrictive case where  $a \leq \frac{1}{8} S_g$ . In this case, when  $a$  is not too large, we get that there are solutions for all phases and that such solutions are uniformly bounded in  $C^{2,\theta}$  for all phases.

**Corollary 0.3** (Critical Case 2). *Let  $(M, g)$  be a smooth compact 3-dimensional Riemannian manifold and  $a > 0$  be a smooth positive function in  $M$ . Let  $q > 0$  and  $\lambda \geq 0$ . Suppose  $a \leq \frac{1}{8} S_g$ , the inequality being strict at least at one point in case the manifold is conformally diffeomorphic to the unit 3-sphere. For any  $\omega \in (-\omega_a, \omega_a)$  there exists a smooth positive solution of (0.2) when  $p = 6$ . Moreover, for any  $\theta \in (0, 1)$ , there exists  $C > 0$  such that  $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$  for all  $\omega \in (-\omega_a, \omega_a)$ , and all  $\mathcal{U} \in \mathcal{S}_6(\omega)$ .*

As an immediate consequence of the  $C^{2,\theta}$ -bounds in the above results we obtain phase stability for standing waves of the Klein-Gordon-Maxwell equations in electrostatic form. Standing waves for the Klein-Gordon-Maxwell equations in electrostatic form can be written as  $S = ue^{i\omega t}$  and they are coupled with a gauge potential  $v$ , where  $(u, v)$  solves (0.2). Roughly speaking, phase stability means that for arbitrary sequences of standing waves  $u_\alpha e^{i\omega_\alpha t}$ , with gauge potentials  $v_\alpha$ , the convergence of the phase  $\omega_\alpha$  in  $\mathbb{R}$  implies the convergence of the amplitude  $u_\alpha$  and of the gauge  $v_\alpha$  in the  $C^2$ -topology. In the subcritical case, it follows from Corollary 0.1 that for any sequence  $(\omega_\alpha)_\alpha$ ,  $\alpha \in \mathbb{N}$ , and for any sequence of standing waves  $(x, t) \rightarrow u_\alpha(x) e^{i\omega_\alpha t}$ , with gauge potentials  $v_\alpha$ , if  $\omega_\alpha \rightarrow \omega$  as  $\alpha \rightarrow +\infty$  and  $|\omega| < \omega_a$ , then, up to a subsequence,  $u_\alpha \rightarrow u$  and  $v_\alpha \rightarrow v$  in  $C^2$  as  $\alpha \rightarrow +\infty$  for some smooth functions  $u$  and  $v$ . By Lemma 4.1,  $u$  and  $v$  are positive and they give rise to another standing wave  $ue^{i\omega t}$  with gauge potential  $v$ . In particular, phase stability in the subcritical case holds true without any condition. By Corollary 0.3, phase stability remains true in the critical case where  $p = 6$  if we assume that  $a \leq \frac{1}{8} S_g$  with the property that the inequality is strict at least at one point if  $(M, g)$  is conformally diffeomorphic to the unit 3-sphere. By Corollary 0.2, assuming  $\lambda > 0$  and  $\text{Osc}(a) < \frac{1}{8} \min_M S_g$ , phase stability also holds true if  $|\omega| < \omega_a$  is sufficiently large such that  $a < \omega^2 + \frac{1}{8} S_g$ . As a remark, phase stability prevents the existence of arbitrarily large amplitude standing waves (see Corollary 1.1 in Section 1).

As a remark it follows from our proofs that the bounds in Corollary 0.1 and Corollary 0.3 are uniform with respect to  $\lambda$  as long as  $\lambda$  stays bounded. In particular, they are uniform with respect to  $\lambda$  as  $\lambda \rightarrow 0$ .

Let  $(S^3, g_0)$  be the unit 3-sphere. Let  $p = 6$ . We have that  $S_{g_0} \equiv 6$ . By our theorem we then get that there are  $C^{2,\theta}$ -bounds for (0.2) in the unit sphere as soon as  $a \leq 3/4$  and the inequality is strict for at least one point in the manifold. On the other hand, by the noncompactness of the conformal group of  $(S^3, g_0)$ , such bounds do not exist anymore when  $a = 3/4$  and  $\lambda = 0$ . In particular, when  $a = 3/4$  and  $\lambda = 0$ , there are sequences of solutions  $(u_\alpha, \frac{1}{q})$  of (0.2) in  $(S^3, g_0)$  with  $p = 6$ ,

$\alpha \in \mathbb{N}$ , which are such that  $u_\alpha \rightharpoonup 0$  weakly in  $H^1$  but not strongly as  $\alpha \rightarrow +\infty$ . Because of the noncompactness of the conformal group of  $(S^3, g_0)$ , the  $C^{2,\theta}$ -bound for (0.2) when  $p = 6$  does not hold true in general when we assume the sole (0.6).

In section 1 we discuss the relation which exists between (0.2), the Klein-Gordon-Maxwell equations, and the Maxwell equations. A related result to Theorem 0.1 is presented in Section 2 when we do not assume a sign on the scalar curvature of the background metric (implicitly required in Theorem 0.1 when  $p = 6$ ) but ask for the potential, balanced by the phase, to be very small (in a non-quantified sense). We prove Theorem 0.1 in sections 3 to 6. The existence part in the theorem is proved in Section 3. The  $C^{2,\theta}$ -bound in the subcritical case  $p \in (2, 6)$  is established in Section 4. The more delicate  $C^{2,\theta}$ -bound in the critical case  $p = 6$  is established in Sections 5 and 6. Theorem 2.1 of Section 2 is proved in Section 7 using the blow-up analysis developed in Sections 5 and 6.

## 1. ACTION INTERPRETATION OF THE SYSTEM AND THE MAXWELL EQUATIONS

We illustrate the background action functional related to our problem and the relation which holds between (0.2), the Maxwell equations, and the Klein-Gordon-Maxwell equations. The model we discuss is a model describing the interactions between matter and electromagnetic fields established, see, for instance, Benci and Fortunato [16], by means of Abelian gauge theories. Formally, the ordinary derivatives  $\partial_t$  and  $\nabla$  in the Klein-Gordon total functional are replaced by gauge covariant derivatives given by the rules  $\partial_t \rightarrow \partial_t + iq\varphi$  and  $\nabla \rightarrow \nabla - iqA$ . Let  $(M, g)$  be a smooth compact Riemannian 3-manifold,  $a > 0$  be a smooth positive function in  $M$ ,  $\lambda \geq 0$ , and  $q > 0$ . We define the Lagrangian densities  $\mathcal{L}_0$  and  $\mathcal{L}_1$  of  $\psi$ ,  $\varphi$ , and  $A$  by

$$\begin{aligned} \mathcal{L}_0(\psi, \varphi, A) &= \frac{1}{2} \left| \left( \frac{\partial}{\partial t} + iq\varphi \right) \psi \right|^2 - \frac{1}{2} |(\nabla - iqA)\psi|^2 \text{ and} \\ \mathcal{L}_1(\varphi, A) &= \frac{1}{2} \left| \frac{\partial A}{\partial t} + \nabla \varphi \right|^2 + \frac{\lambda}{2} |\varphi|^2 - \frac{1}{2} |\nabla \times A|^2, \end{aligned} \quad (1.1)$$

where  $\nabla \times$  denotes the curl operator defined thanks to the Hodge dual  $\star$  when  $M$  is orientable. In this model,  $\psi$  is a matter field,  $(A, \varphi)$  are gauge potentials representing the electromagnetic field  $(E, H)$  it generates as in (1.5),  $q$  is a nonzero coupling constant, representing the electric charge, and  $\lambda \geq 0$  is a coercivity constant which generates phase compensation. Then  $\mathcal{L}_1$  is a 0-order perturbation, by  $\frac{\lambda}{2}\varphi^2$ , of the standard electromagnetic Lagrangian density

$$\begin{aligned} \mathcal{L}_1^0 &= \frac{1}{2} \left( \left| \frac{\partial A}{\partial t} + \nabla \varphi \right|^2 - |\nabla \times A|^2 \right) \\ &= \frac{1}{2} (|E|^2 - |H|^2) \end{aligned}$$

associated to the electromagnetic field  $(E, H)$  given by (1.5). We let  $\mathcal{S}$  be the total action functional for  $\psi$ ,  $\varphi$ ,  $A$  defined by

$$\mathcal{S}(\psi, \varphi, A) = \int \int (\mathcal{L}_0 + \mathcal{L}_1 - W) dv_g dt, \quad (1.2)$$

where  $\mathcal{L}_0$  and  $\mathcal{L}_1$  are as in (1.1), and  $W$  is a function of  $\psi$  given by

$$W(\psi) = \frac{a}{2}|\psi|^2 - \frac{1}{p}|\psi|^p . \quad (1.3)$$

Writing  $\psi$  in polar form as

$$\psi(x, t) = u(x, t)e^{iS(x, t)}$$

for  $u \geq 0$  and  $S \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ , the total action functional  $\mathcal{S}$  given by (1.2) is written as

$$\begin{aligned} \mathcal{S}(u, S, \varphi, A) = & \frac{1}{2} \int \int \left( \left( \frac{\partial u}{\partial t} \right)^2 - |\nabla u|^2 - au^2 \right) dv_g dt + \frac{1}{p} \int \int u^p dv_g dt \\ & + \frac{1}{2} \int \int \left( \left( \frac{\partial S}{\partial t} + q\varphi \right)^2 - |\nabla S - qA|^2 \right) u^2 dv_g dt \\ & + \frac{1}{2} \int \int \left( \left| \frac{\partial A}{\partial t} + \nabla \varphi \right|^2 + \lambda \varphi^2 - |\nabla \times A|^2 \right) dv_g dt . \end{aligned}$$

Taking the variation of  $\mathcal{S}$  with respect to  $u$ ,  $S$ ,  $\varphi$ , and  $A$ , we get four equations which are written as

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta_g u + au = u^{p-1} + \left( \left( \frac{\partial S}{\partial t} + q\varphi \right)^2 - |\nabla S - qA|^2 \right) u \\ \frac{\partial}{\partial t} \left( \left( \frac{\partial S}{\partial t} + q\varphi \right) u^2 \right) - \nabla \cdot \left( (\nabla S - qA) u^2 \right) = 0 \\ -\nabla \cdot \left( \left( \frac{\partial A}{\partial t} + \nabla \varphi \right) + \lambda \varphi + q \left( \frac{\partial S}{\partial t} + q\varphi \right) u^2 \right) = 0 \\ \nabla \times (\nabla \times A) + \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial t} + \nabla \varphi \right) = q (\nabla S - qA) u^2 . \end{cases} \quad (1.4)$$

Now let

$$\begin{aligned} E &= - \left( \frac{\partial A}{\partial t} + \nabla \varphi \right) , \quad H = \nabla \times A , \\ \rho &= - \left( \frac{\partial S}{\partial t} + q\varphi \right) qu^2 - \lambda \varphi , \quad \text{and } j = (\nabla S - qA) qu^2 . \end{aligned} \quad (1.5)$$

Then the two last equations in (1.4) give rise to the second couple of the Maxwell equations with respect to a matter distribution whose charge and current density are respectively  $\rho$  and  $j$ , namely

$$\nabla \cdot E = \rho \text{ and} \quad (1.6)$$

$$\nabla \times H - \frac{\partial E}{\partial t} = j , \quad (1.7)$$

while the two first equations in (1.5) give rise to the first couple of the Maxwell equations:

$$\nabla \times E + \frac{\partial H}{\partial t} = 0 \text{ and} \quad (1.8)$$

$$\nabla \cdot H = 0 . \quad (1.9)$$

In addition, the first equation in (1.4) gives rise to the matter equation

$$\frac{\partial^2 u}{\partial t^2} + \Delta_g u + au = u^{p-1} + \frac{\kappa}{q^2 u^3} , \quad (1.10)$$

where  $\kappa = (\rho + \lambda\varphi)^2 - j^2$ , while the second equation in (1.4) gives rise to the charge continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0 \quad (1.11)$$

if we assume that  $\varphi = \varphi(x)$ . In particular, we recover with (1.4) the Maxwell equations (1.6)–(1.9) together with a Klein-Gordon type equation (1.10). The system (1.4) is referred to as the Klein-Gordon-Maxwell system. Suppose now that  $S = -\omega t$ ,  $\omega$  real, that  $u = u(x)$ , and that  $A = 0$ . Then we are in the electrostatic form of the above equations and we search for standing waves for these equations. In such a setting, the second and the fourth equations in (1.4) are automatically satisfied, while the first and the third equations in (1.4) become

$$\begin{cases} \Delta_g u + au = u^{p-1} + (q\varphi - \omega)^2 u \\ \Delta_g \varphi + \lambda\varphi + q(q\varphi - \omega)u^2 = 0. \end{cases} \quad (1.12)$$

In particular, letting  $\varphi = \omega v$ , we recover our original system (0.2).

A direct consequence of our result concerns the amplitude of standing waves for the electrostatic form of the Klein-Gordon-Maxwell system (1.4). It illustrates the idea that phase stability prevents the existence of arbitrarily large amplitude standing waves. More precisely, the following corollary is a direct consequence of the results stated in the introduction.

**Corollary 1.1** (On the amplitude of standing waves). *Let  $(M, g)$  be a smooth compact 3-dimensional Riemannian manifold and  $a > 0$  be a smooth positive function in  $M$ . Let  $q > 0$  and  $p \in (2, 6]$ . Assume that one of the three following assumptions hold true:*

- (i)  $\lambda \geq 0$ ,  $p < 6$ , or
- (ii)  $\lambda > 0$ ,  $p = 6$ ,  $\text{Osc}(a) < \frac{1}{8} \min_M S_g$ , or
- (iii)  $\lambda \geq 0$ ,  $p = 6$ ,  $a \leq \frac{1}{8} S_g$  and the inequality is strict at one point if  $(M, g)$  is conformally diffeomorphic to the unit 3-sphere.

*Then, for any  $\theta \in (0, 1)$ , there exists  $C > 0$  such that  $\|u\|_{C^{2,\theta}} \leq C$  for all standing waves  $ue^{-i\omega t}$  of (1.4) in its electrostatic form  $A = 0$ , all  $\omega \in (-\omega_a, \omega_a)$  in case (i), all  $\omega \in K_0(\omega_a - \varepsilon)$  in case (ii), where  $\varepsilon > 0$  is suitably chosen, and all  $\omega \in (-\omega_a, \omega_a)$  in case (iii). Moreover, there also holds that  $\|\varphi\|_{C^{2,\theta}} \leq C$  for the gauge potential  $\varphi$ .*

Klein-Gordon-Maxwell systems have been investigated by Bechouche, Mauser and Selberg [15], Choquet-Bruhat [28], Deumens [29], Eardley and Moncrief [39], Klainerman and Machedon [48], Machedon and Sterbenz [55], Masmoudi and Nakanishi [57, 58], Petrescu [60], Rodnianski and Tao [62], and Tao [71].

Existence of solutions and semiclassical limits for systems like (0.2), in Euclidean space, for subcritical nonlinear terms, have been investigated by Ambrosetti and Ruiz [2], D’Aprile and Mugnai [5, 6], D’Aprile and Wei [7, 8], D’Avenia and Pisani [9], D’Avenia, Pisani and Siciliano [10, 11], Azzollini, D’Avenia and Pomponio [12], Azzollini and Pomponio [13, 14], Benci and Fortunato [16, 18, 19], Bonanno [21], Cassani [27], Ianni and Vaira [46], Long [54], Mugnai [59], and Ruiz [63].

Existence and nonexistence of a priori estimates for critical elliptic Schrödinger equations on manifolds have been investigated by Berti-Malchiodi [20], Brendle [22, 23], Brendle and Marques [24], Druet [30, 31], Druet and Hebey [33, 34], Druet,

Hebey, and Vétois [37], Druet and Laurain [38], Khuri, Marques and Schoen [47], Li and Zhang [50, 51], Li and Zhu [53], Marques [56], Schoen [66, 67], and Vétois [73]. In the subcritical case, a priori estimates for subcritical Schrödinger equations goes back to the seminal work by Gidas and Spruck [40]. The above lists are not exhaustive.

The positive mass theorem in general relativity, that we use below in this paper, was established in Schoen and Yau [68]. We refer also to Schoen and Yau [69, 70] and Witten [74].

## 2. ONE MORE ESTIMATE

Theorem 0.1 in the critical case  $p = 6$  implicitly requires that the scalar curvature of the background manifold is positive. When this is not the case, there are still several situations where (0.2) possesses positive solutions. Such situations are, for instance, easy to obtain by requiring  $a$  to be  $G$ -invariant, where  $G$  is a subgroup of the isometry group of  $G$  having no finite orbits, and by using the improved Sobolev embeddings in Hebey and Vaugon [45] (see also Hebey [41]). Using part of the analysis developed to prove Theorem 0.1 we may independently get a priori bounds for the set of solutions when the mass potential, balanced by the phase, is sufficiently small (in a non-quantified sense). In particular, we can prove that the following theorem holds true. Given  $R > 0$  we let  $B_R^{0,1}$  be the set of smooth functions  $a \in C^\infty(M)$  such that  $\|a\|_{C^{0,1}} \leq R$ .

**Theorem 2.1.** *Let  $(M, g)$  be a smooth compact 3-dimensional Riemannian manifold,  $\lambda \geq 0$ ,  $R > 0$ ,  $q > 0$ , and  $\theta \in (0, 1)$ . There exist  $\varepsilon, C > 0$ , depending only on  $M, g, R, \lambda, q$ , and  $\theta$ , such that for any  $a \in B_R^{0,1}$ ,  $a > 0$ , and any  $\omega \in (-\omega_a, \omega_a)$ , if*

*(i)  $a - \omega^2 < \varepsilon$  and  $\lambda > 0$ , or*

*(ii)  $a < \varepsilon$  and  $\lambda \geq 0$ ,*

*then  $\|U\|_{C^{2,\theta}} \leq C$  for all  $U \in \mathcal{S}_6(\omega')$ , all  $\omega' \in (-\omega_a, -\omega] \cup [\omega, \omega_a)$  in case (i), and all  $\omega' \in (-\omega_a, \omega_a)$  in case (ii).*

A similar corollary to Corollary 1.1 can be derived from the above theorem. In particular, we prevent the existence of arbitrarily large amplitude standing waves for the Klein-Gordon-Maxwell system in electrostatic form when either (i) or (ii) is assumed to hold. The  $C^{0,1}$ -bound on  $a$  in Theorem 2.1 can be lowered. We require the  $C^{0,1}$ -bound to get  $C^{2,\theta}$ -estimates on solutions without restrictions on  $\theta$ . Theorem 2.1 is proved in Section 7.

## 3. VARIATIONAL ANALYSIS AND THE EXISTENCE PART OF THEOREM 0.1

Solutions  $(u, v)$  of (0.2) are critical points of a functional  $S$  defined on  $H^1 \times H^1$  where the  $H^1$ -norms of  $u$  and  $v$  compete one with another. More precisely,

$$\begin{aligned} S(u, v) = & \frac{1}{2} \int_M |\nabla u|^2 dv_g - \frac{\omega^2}{2} \int_M |\nabla v|^2 dv_g + \frac{1}{2} \int_M a u^2 dv_g \\ & - \frac{\omega^2 \lambda}{2} \int_M v^2 dv_g - \frac{1}{p} \int_M u^p dv_g - \frac{\omega^2}{2} \int_M u^2 (1 - qv)^2 dv_g. \end{aligned} \quad (3.1)$$

As is easily checked,  $S(0, v) < 0$  if  $v$  is nonconstant and, for any  $R > 0$ , there exists  $u \in H^1$  such that  $\|u\|_{H^1} \geq R$  and  $S(tu, 0) > 0$  for all  $t \in (0, 1]$ . In order to overcome the problems caused by the competition between  $u$  and  $v$  in  $S$ , following

the very nice idea in Benci and Fortunato [16], we introduce the map  $\Phi : H^1 \rightarrow H^1$  defined by the equation

$$\Delta_g \Phi(u) + (\lambda + q^2 u^2) \Phi(u) = qu^2. \quad (3.2)$$

It follows from standard variational arguments that  $\Phi$  is well-defined in  $H^1$  as soon as  $\lambda > 0$ . Noting that  $u^2 \Phi(u) \in L^2$  since  $u$  and  $\Phi(u)$  are in  $H^1$ , it follows from (3.2) that  $\Phi(u) \in H^2$ . We further get with (3.2) that  $\Phi(u) \in H^{2.3}$ . Elementary though useful properties of  $\Phi$  are as follows.

**Lemma 3.1.** *The map  $\Phi : H^1 \rightarrow H^1$  is  $C^1$  and its differential  $D\Phi(u) = V_u$  at  $u$  is the map defined by*

$$\Delta_g V_u(h) + (\lambda + q^2 u^2) V_u(h) = 2qu(1 - q\Phi(u))h \quad (3.3)$$

for all  $h \in H^1$ . Moreover, it holds that

$$0 \leq \Phi(u) \leq \frac{1}{q} \quad (3.4)$$

for all  $u \in H^1$ .

*Proof of Lemma 3.1.* Let  $u \in H^1$ . It is clear from the maximum principle that  $\Phi(u) \geq 0$ . Noting that

$$\Delta_g \left( \frac{1}{q} - \Phi(u) \right) + (\lambda + q^2 u^2) \left( \frac{1}{q} - \Phi(u) \right) = \frac{\lambda}{q} \quad (3.5)$$

it also follows from the maximum principle that  $\Phi(u) \leq \frac{1}{q}$ , and this proves (3.4). Given  $h \in H^1$ , we compute

$$\begin{aligned} & \Delta_g (\Phi(u+h) - \Phi(u)) + (\lambda + q^2 u^2) (\Phi(u+h) - \Phi(u)) \\ &= q(1 - q\Phi(u+h))(h^2 + 2uh). \end{aligned} \quad (3.6)$$

Multiplying (3.6) by  $\Phi(u+h) - \Phi(u)$ , and integrating over  $M$ , by the coercivity of  $\Delta_g + \lambda$ , by the Sobolev embedding theorem, by Hölder's inequality, and by (3.4), we get that

$$\|\Phi(u+h) - \Phi(u)\|_{H^1} \leq C(u) \|h\|_{H^1} (1 + \|h\|_{H^1})$$

for all  $h \in H^1$ , where  $C(u) > 0$  is independent of  $h$ . In particular,  $\Phi$  is continuous. Also we compute,

$$\begin{aligned} & \Delta_g (\Phi(u+h) - \Phi(u) - V_u(h)) + (\lambda + q^2 u^2) (\Phi(u+h) - \Phi(u) - V_u(h)) \\ &= qh^2 - q^2 \Phi(u+h)h^2 + 2q^2 u (\Phi(u) - \Phi(u+h))h. \end{aligned} \quad (3.7)$$

Multiplying (3.7) by  $\Phi(u+h) - \Phi(u) - V_u(h)$  and integrating over  $M$ , by the coercivity of  $\Delta_g + \lambda$ , by the Sobolev embedding theorem, by Hölder's inequality, and by (3.4), we get that

$$\|\Phi(u+h) - \Phi(u) - V_u(h)\|_{H^1} \leq C(u) \|h\|_{H^1} (\|h\|_{H^1} + \|\Phi(u+h) - \Phi(u)\|_{H^1})$$

for all  $h \in H^1$ , where  $C(u) > 0$  is independent of  $h$ . Since  $\Phi$  is continuous, the differentiability of  $\Phi$  at  $u$ , as well as the fact that  $D\Phi(u) = V_u$ , follow from this estimate. The continuity of  $u \rightarrow V_u$  can be proved with similar arguments. This proves the lemma.  $\square$



Coming back to (3.6), and the procedure described after (3.6), it holds true that

$$\|\Phi(v) - \Phi(u)\|_{H^1} \leq C(u, v) \|v - u\|_{L^3} \quad (3.8)$$

for all  $u, v \in H^1$ , where  $C(u, v) = C(\|u\|_{L^3} + \|v\|_{L^3})$  and  $C > 0$  is independent of  $u$  and  $v$ . We can also write that

$$\|V_v(h) - V_u(h)\|_{H^1} \leq C(u, v) \|v - u\|_{L^3} \|h\|_{L^6} \quad (3.9)$$

for all  $u, v, h \in H^1$ , where  $C(u, v) = C(1 + \|u\|_{L^3}^2 + \|v\|_{L^3}^2)$  and  $C > 0$  is independent of  $u$  and  $v$ .

**Lemma 3.2.** *The map  $\Theta : H^1 \rightarrow \mathbb{R}$  given by*

$$\Theta(u) = \frac{1}{2} \int_M (1 - q\Phi(u)) u^2 dv_g \quad (3.10)$$

*is  $C^1$  and, for any  $u \in H^1$ ,*

$$D\Theta(u).(h) = \int_M (1 - q\Phi(u))^2 u h dv_g \quad (3.11)$$

*for all  $h \in H^1$ .*

*Proof of Lemma 3.2.* It follows from Lemma 3.1 that  $\Theta$  is  $C^1$ . By (3.2),

$$\Theta(u) = \frac{1}{2} \int_M (|\nabla \Phi(u)|^2 + \lambda \Phi(u)^2) dv_g + \frac{1}{2} \int_M (1 - q\Phi(u))^2 u^2 dv_g,$$

and we also have that  $\frac{\partial H}{\partial \Phi}(u, \Phi(u)) = 0$ , where

$$H(u, \Phi) = \frac{1}{2} \int_M (|\nabla \Phi|^2 + \lambda \Phi^2) dv_g + \frac{q^2}{2} \int_M u^2 \Phi^2 dv_g - q \int_M u^2 \Phi dv_g.$$

Noting that

$$\Theta(u) = H(u, \Phi(u)) + \frac{1}{2} \int_M u^2 dv_g,$$

we get that (3.11) holds true. This ends the proof of the lemma.  $\square$

At this point we define the functional  $I_p : H^1 \rightarrow \mathbb{R}$  by

$$\begin{aligned} I_p(u) &= \frac{1}{2} \int_M |\nabla u|^2 dv_g + \frac{1}{2} \int_M a u^2 dv_g - \frac{1}{p} \int_M (u^+)^p dv_g \\ &\quad - \frac{\omega^2}{2} \int_M (1 - q\Phi(u)) (u^+)^2 dv_g, \end{aligned} \quad (3.12)$$

where  $u^+ = \max(u, 0)$ , and  $p \in (2, 6]$ . By Lemma 3.1 we get that  $I_p$  is  $C^1$  and if  $\hat{\Theta}$  is given by

$$\hat{\Theta}(u) = \int_M (1 - q\Phi(u)) (u^+)^2 dv_g \quad (3.13)$$

then, for any  $u \in H^1$ ,

$$D\hat{\Theta}(u).(h) = 2 \int_M (1 - q\Phi(u)) u^+ h dv_g - q \int_M (u^+)^2 V_u(h) dv_g \quad (3.14)$$

for all  $h \in H^1$ , where  $V_u$  is as in (3.3). First we prove the existence part of Theorem 0.1 when  $p \in (2, 6)$ . For this aim we use the mountain pass lemma, as stated in Ambrosetti and Rabinowitz [1], that we apply to the functional  $I_p$  defined in (3.12).

*Proof of the Existence Part in Theorem 0.1 when  $p \in (2, 6)$ .* Suppose first that  $\lambda = 0$ . Since  $p < 6$  and  $a > 0$  there exists  $u$  smooth and positive such that

$$\Delta_g u + au = u^{p-1}.$$

The existence of  $u$  easily follows from standard variational arguments. Then  $(u, \frac{1}{q})$  solve (0.2). From now on we assume  $\lambda > 0$ . It is easily checked that  $I_p(0) = 0$  and that for  $u_0$  arbitrarily given in  $H^1$ , with  $u_0^+ \neq 0$ , there holds that  $I_p(tu_0) \rightarrow -\infty$  as  $t \rightarrow +\infty$ . Moreover, since we assumed that  $\omega \in (-\omega_a, \omega_a)$ , we get from (3.4) and the Sobolev embedding theorem that

$$\begin{aligned} I_p(u) &\geq \frac{1}{2} \int_M |\nabla u|^2 dv_g + \frac{1}{2} \int_M (a - \omega^2) u^2 dv_g - \frac{1}{p} \int_M |u|^p dv_g \\ &\geq C_1 \|u\|_{H^1}^2 - C_2 \|u\|_{H^1}^p, \end{aligned} \quad (3.15)$$

where  $C_1, C_2 > 0$ . In particular, it follows from (3.15) that there exist  $\delta > 0$  and  $C > 0$  such that  $I_p(u) \geq C$  for all  $u \in H^1$  which satisfy  $\|u\|_{H^1} = \delta$ . Let  $T_0 = T(u_0)$  be such that  $I_p(T_0 u_0) < 0$ . Let  $c_p = c_p(u_0)$  be given by

$$c_p = \inf_{P \in \mathcal{P}} \max_{u \in P} I_p(u), \quad (3.16)$$

where  $\mathcal{P}$  denotes the class of continuous paths joining 0 to  $T_0 u_0$ . By the mountain pass lemma, see Ambrosetti and Rabinowitz [1], there exists  $(u_\alpha)_\alpha$  in  $H^1$  such that  $I_p(u_\alpha) \rightarrow c_p$  and  $DI_p(u_\alpha) \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Writing that  $I_p(u_\alpha) = c_p + o(1)$  and that  $DI_p(u_\alpha).(u_\alpha) = o(\|u_\alpha\|_{H^1})$  we get that

$$\begin{aligned} &\frac{1}{2} \int_M (|\nabla u_\alpha|^2 + a u_\alpha^2) dv_g \\ &= \frac{1}{p} \int_M (u_\alpha^+)^p dv_g + c_p + \frac{\omega^2}{2} \hat{\Theta}(u_\alpha) + o(1), \text{ and} \\ &\frac{1}{2} \int_M (|\nabla u_\alpha|^2 + a u_\alpha^2) dv_g \\ &= \frac{1}{2} \int_M (u_\alpha^+)^p dv_g + \frac{\omega^2}{4} D\hat{\Theta}(u_\alpha).(u_\alpha) + o(\|u_\alpha\|_{H^1}), \end{aligned} \quad (3.17)$$

where  $\hat{\Theta}$  is as in (3.13). By (3.3) and (3.14), for any  $u \in H^1$ ,

$$\Delta_g (V_u(u) - 2\Phi(u)) + (\lambda + q^2 u^2) (V_u(u) - 2\Phi(u)) = -2q^2 u^2 \Phi(u) \leq 0.$$

By the maximum principle and (3.4) we then get that

$$0 \leq V_u(u) \leq 2\Phi(u) \leq \frac{2}{q} \quad (3.18)$$

for all  $u \in H^1$ . In particular, thanks to (3.14) and (3.18), we get that

$$|D\hat{\Theta}(u).(u)| \leq C \int_M (u^+)^2 dv_g \quad (3.19)$$

for all  $u \in H^1$ , where  $C > 0$  is independent of  $u$ . Subtracting the second equation in (3.17) to the first, thanks to (3.18), it follows that

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{p}\right) \|u_\alpha^+\|_{L^p}^p &\leq c_p + o(1) + C \|u_\alpha^+\|_{L^2}^2 + o(\|u_\alpha\|_{H^1}) \\ &\leq c_p + o(1) + C' \|u_\alpha^+\|_{L^p}^2 + o(\|u_\alpha\|_{H^1}) \end{aligned} \quad (3.20)$$

for all  $\alpha$ , where  $C, C' > 0$  are independent of  $\alpha$ . By (3.17) and (3.20), since  $p > 2$ , the sequence  $(u_\alpha)_\alpha$  is bounded in  $H^1$ . Since  $p < 6$  we may then assume that there exists  $u_p \in H^1$  such that, up to a subsequence,  $u_\alpha \rightharpoonup u_p$  in  $H^1$  and  $u_\alpha \rightarrow u_p$  in  $L^p \cap L^3$  as  $\alpha \rightarrow +\infty$ . Since  $c_p > 0$ , it is clear from (3.17) that  $u_p \neq 0$ . For any  $\varphi \in H^1$ ,  $DI_p(u_\alpha).(\varphi) = o(1)$ . Letting  $\alpha \rightarrow +\infty$  in this equation, thanks to (3.8), (3.9), and (3.11), it follows that for any  $\varphi \in H^1$ ,

$$\begin{aligned} & \int_M (\nabla u_p \nabla \varphi) dv_g + \int_M a u_p \varphi dv_g \\ &= \int_M (u_p^+)^{p-1} \varphi dv_g + \omega^2 \int_M (1 - q\Phi(u_p)) u_p^+ \varphi dv_g \\ & \quad - \frac{q\omega^2}{2} \int_M V_{u_p}(\varphi) (u_p^+)^2 dv_g. \end{aligned} \quad (3.21)$$

Noting that  $V_{u_p}(u_p^-) \leq 0$ , where  $u^- = \max(-u, 0)$ , we get from (3.21) that

$$\int_M ((\nabla u_p \nabla u_p^-) + a u_p u_p^-) dv_g \geq 0$$

and it follows that  $u_p^- \equiv 0$ . Hence  $u_p \geq 0$  in  $M$ . By Lemma 3.2 and (3.21) we can then write that for any  $\varphi \in H^1$ ,

$$\begin{aligned} & \int_M (\nabla u_p \nabla \varphi) dv_g + \int_M a u_p \varphi dv_g \\ &= \int_M u_p^{p-1} \varphi dv_g + \omega^2 \int_M (1 - q\Phi(u_p))^2 u_p \varphi dv_g. \end{aligned} \quad (3.22)$$

In particular, by (3.22), we get that  $(u_p, \Phi(u_p))$  solves (0.2). By the maximum principle and elliptic regularity we get that  $u_p > 0$ ,  $\Phi(u_p) > 0$ , and  $u_p$  and  $\Phi(u_p)$  are smooth. This ends the proof of the existence part in Theorem 0.1 when  $p \in (2, 6)$ .  $\square$

As already mentionned, it follows from (3.17) that the sequence  $(u_\alpha)_\alpha$  is bounded in  $H^1$ . By (3.8) and (3.9) we then get that

$$D\hat{\Theta}(u_\alpha).(u_\alpha) \rightarrow D\hat{\Theta}(u_p).(u_p)$$

as  $\alpha \rightarrow +\infty$  since  $u_\alpha \rightarrow u_p$  in  $L^p \cap L^3$ . Also there holds that  $\int_M a u_\alpha^2 dv_g \rightarrow \int_M a u_p^2 dv_g$  and  $\int_M (u_\alpha^+)^p dv_g \rightarrow \int_M u_p^p dv_g$  as  $\alpha \rightarrow +\infty$ . By (3.21),

$$\frac{1}{2} \int_M (|\nabla u_p|^2 + a u_p^2) dv_g = \frac{1}{2} \int_M u_p^p dv_g + \frac{\omega^2}{4} D\hat{\Theta}(u_p).(u_p).$$

Coming back to the second equation in (3.17) we get that

$$\int_M |\nabla u_\alpha|^2 dv_g \rightarrow \int_M |\nabla u_p|^2 dv_g$$

as  $\alpha \rightarrow +\infty$  and it follows that  $u_\alpha \rightarrow u_p$  in  $H^1$  as  $\alpha \rightarrow +\infty$ . By the first equation in (3.17), since by (3.8) we can write that  $\hat{\Theta}(u_\alpha) \rightarrow \hat{\Theta}(u_p)$  as  $\alpha \rightarrow +\infty$ , we get that

$$I_p(u_p) = c_p, \quad (3.23)$$

where  $c_p$  is as in (3.16).

At this point it remains to prove the existence part of Theorem 0.1 when  $p = 6$ . For this aim we use the existence of subcritical solutions we just obtained and the developments in Schoen [64] based on the positive mass theorem of Schoen

and Yau [68] (see also Schoen and Yau [69, 70] and Witten [74]). Let  $G$  be the Green's function of the conformal Laplacian  $\Delta_g + \frac{1}{8}S_g$ . Let  $x_0$  be given in  $M$  and  $G(x) = G(x_0, x)$ . In geodesic normal coordinates,

$$G(x) = \frac{1}{\omega_2|x|} + A + \alpha(x) , \quad (3.24)$$

where  $\omega_2$  is the volume of the unit 2-sphere, and  $\alpha(x) = O(|x|)$ . Noting that  $\hat{g} = G^4g$  is scalar flat and asymptotically Euclidean, it is a consequence of the positive mass theorem that  $A \geq 0$  and  $A = 0$  if and only if  $(M, g)$  is conformally diffeomorphic to the unit sphere. Following Schoen [64], we let  $\rho_0 > 0$  be a small radius and  $\varepsilon_0 > 0$  to be chosen small relative to  $\rho_0$ . Let also  $\psi$  be a piecewise smooth decreasing function of  $|x|$  such that  $\psi(x) = 1$  for  $|x| \leq \rho_0$ ,  $\psi(x) = 0$  for  $|x| \geq 2\rho_0$ , and  $|\nabla\psi| \leq \rho_0^{-1}$  for  $\rho_0 \leq |x| \leq 2\rho_0$ . We define  $u_\varepsilon$ ,  $\varepsilon > 0$ , by

$$\begin{cases} u_\varepsilon(x) = \left( \frac{\varepsilon}{\varepsilon^2 + d_g(x_0, x)^2} \right)^{1/2} & \text{for } d_g(x_0, x) \leq \rho_0 , \\ u_\varepsilon(x) = \varepsilon_0 (G(x) - \psi(x)\alpha(x)) & \text{for } \rho_0 \leq d_g(x_0, x) \leq 2\rho_0 , \\ u_\varepsilon(x) = \varepsilon_0 G(x) & \text{for } d_g(x_0, x) \geq 2\rho_0 . \end{cases} \quad (3.25)$$

and require that

$$\varepsilon_0 \left( \frac{1}{\omega_2\rho_0} + A \right) = \sqrt{\frac{\varepsilon}{\varepsilon^2 + \rho_0^2}} .$$

Then, see Schoen [64], since  $A > 0$  if  $(M, g)$  is not conformally diffeomorphic to  $(S^3, g_0)$ , we get that

$$\frac{\int_M (|\nabla u_\varepsilon|^2 + \frac{1}{8}S_g u_\varepsilon^2) dv_g}{\left( \int_M u_\varepsilon^6 dv_g \right)^{1/3}} < \frac{1}{K_3^2} \quad (3.26)$$

for  $\varepsilon \ll 1$ , when  $(M, g)$  is not conformally diffeomorphic to  $(S^3, g_0)$ , where  $K_3$  is the sharp constant in the Euclidean Sobolev inequality

$$\left( \int_{\mathbb{R}^3} |u|^6 dx \right)^{1/3} \leq K_3^2 \int_{\mathbb{R}^3} |\nabla u|^2 dx . \quad (3.27)$$

Also there holds that

$$\int_M u_\varepsilon^6 dv_g = \int_{\mathbb{R}^3} \left( \frac{1}{1 + |x|^2} \right)^3 dx + o(1) . \quad (3.28)$$

Now we split the proof of the existence part of Theorem 0.1 when  $p = 6$  in two cases. In the first case we assume that  $(M, g)$  is not conformally diffeomorphic to the unit 3-sphere. In the second case we assume that  $(M, g)$  is conformally diffeomorphic to the unit 3-sphere.

We use in what follows phase compensation for electrostatic Klein-Gordon-Maxwell systems. Phase compensation follows from the subcritical nature of the second equation in (0.2). It is a key tool in studying (0.2) and it can be explained in naive terms in the following way: if  $(u, v)$  solves (0.2), and  $u$  is small in  $L^{p'}$ , for  $p'$  sufficiently large but still in the subcritical range, then, by Sobolev embeddings,  $v$  needs to be small in  $L^\infty$ , and the potential like term  $a - \omega^2(qv - 1)^2$  in the nonlinear Schrödinger equation in (0.2) approaches  $a - \omega^2$ . In particular, the nonlinear Schrödinger equation in (0.2), and its variational formulations, approach the static isolated nonlinear model Schrödinger equation with unknown function  $u$ , potential  $a - \omega^2$ , and nonlinear term  $u^{p-1}$ .

*Proof of the Existence Part in Theorem 0.1 when  $p = 6$ . Case 1.* We assume that  $(M, g)$  is not conformally diffeomorphic to the unit 3-sphere. Suppose first that  $\lambda = 0$  or that  $\omega = 0$ . By (0.6) and (3.26) we then get that  $a \leq \frac{1}{8}S_g$  and that

$$\frac{\int_M (|\nabla u_\varepsilon|^2 + au_\varepsilon^2) dv_g}{(\int_M u_\varepsilon^6 dv_g)^{1/3}} < \frac{1}{K_3^2}$$

for  $\varepsilon \ll 1$ . In particular, see for instance Aubin [3, 4], there exists  $u$  smooth and positive such that

$$\Delta_g u + au = u^5.$$

Then  $(u, \frac{1}{q})$  solves (0.2) if  $\lambda = 0$ , and  $(u, \Phi(u))$  solves (0.2) if  $\lambda > 0$ . From now on we assume that  $\lambda > 0$  and  $\omega^2 > 0$ . Let  $(\varepsilon_\alpha)_\alpha$  be a sequence of positive real numbers such that  $\varepsilon_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Let  $u_\alpha = u_{\varepsilon_\alpha}$ . By (3.28), there exists  $T > 0$ , independent of  $\alpha$ , such that  $I_6(Tu_\alpha) < 0$  for all  $\alpha \gg 1$ . By (3.28), noting that  $u_\alpha \rightarrow 0$  a.e. as  $\alpha \rightarrow +\infty$ , we get that  $u_\alpha \rightarrow 0$  in  $L^q$  for all  $q < 6$  as  $\alpha \rightarrow +\infty$ . Let  $(t_\alpha)_\alpha$  be any sequence in  $[0, T]$ . We have that  $\Phi(0) = 0$  and thus, by (3.8), there holds that  $\Phi(t_\alpha u_\alpha) \rightarrow 0$  in  $H^1$  as  $\alpha \rightarrow +\infty$ . By (3.2) and (3.4) we then get that  $\Phi(t_\alpha u_\alpha) \rightarrow 0$  in  $H^{2,q}$  for all  $q < 3$  as  $\alpha \rightarrow +\infty$ . In particular,  $\Phi(t_\alpha u_\alpha) \rightarrow 0$  in  $L^\infty$  as  $\alpha \rightarrow +\infty$ , and we can write that

$$\max_{0 \leq t \leq T} \|\Phi(tu_\alpha)\|_{L^\infty} \rightarrow 0 \quad (3.29)$$

as  $\alpha \rightarrow +\infty$ . Since  $k\lambda < 1$ , we get with (3.29) that for any  $\alpha \gg 1$ , and any  $t \in [0, T]$ ,

$$\int_M (1 - q\Phi(tu_\alpha)) u_\alpha^2 dv_g \geq k\lambda \int_M u_\alpha^2 dv_g \quad (3.30)$$

Let  $\mathcal{F}_6$  be the functional defined in  $H^1$  by

$$\mathcal{F}_6(u) = \frac{1}{2} \int_M |\nabla u|^2 dv_g + \frac{1}{16} \int_M S_g u^2 dv_g - \frac{1}{6} \int_M |u|^6 dv_g. \quad (3.31)$$

By (0.6) and (3.30),

$$\max_{0 \leq t \leq T} I_6(tu_\alpha) \leq \max_{0 \leq t \leq T} \mathcal{F}_6(tu_\alpha) \quad (3.32)$$

for all  $\alpha \gg 1$ . Fix  $u_0 = u_\alpha$  for  $\alpha \gg 1$ , sufficiently large such that (3.26) holds true, and let  $T_0 = T$ . For  $\varepsilon > 0$  sufficiently small,  $I_p(T_0 u_0) < 0$  and

$$\max_{0 \leq t \leq T_0} I_p(tu_0) \leq (1 + \delta_\varepsilon) \max_{0 \leq t \leq T_0} I_6(tu_0) \quad (3.33)$$

for all  $p \in (6 - \varepsilon, 6)$ , where  $\delta_\varepsilon > 0$  is such that  $\delta_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . As is easily checked, differentiating  $\mathcal{F}_6(tu_0)$  with respect to  $t$ , we get that

$$\max_{0 \leq t \leq T_0} \mathcal{F}_6(tu_0) \leq \left( \frac{1}{2} - \frac{1}{6} \right) \left( \frac{\int_M (|\nabla u_0|^2 + \frac{1}{8} S_g u_0^2) dv_g}{(\int_M u_0^6 dv_g)^{1/3}} \right)^{3/2}. \quad (3.34)$$

We have that

$$c_p \leq \max_{0 \leq t \leq T_0} I_p(tu_0),$$

where  $c_p$  is as in (3.16). By (3.26), (3.32), (3.33), and (3.34), we then get that there exists  $\delta_0 > 0$  such that for  $0 < \varepsilon \ll 1$  sufficiently small,

$$\delta_0 \leq c_p \leq \frac{1}{3K_3^3} - \delta_0 \quad (3.35)$$

for all  $p \in (6 - \varepsilon, 6)$ . Let  $u_p$ ,  $p < 6$ , be the solution obtained in the subcritical part of the proof of existence. We have, see (3.23), that  $I_p(u_p) = c_p$  and also that  $I'_p(u_p) = 0$ . In particular,

$$\frac{1}{2} \int_M (|\nabla u_p|^2 + a u_p^2) dv_g = c_p + \frac{1}{p} \int_M u_p^p dv_g + \frac{\omega^2}{2} \int_M (1 - q\Phi(u_p)) u_p^2 dv_g, \quad (3.36)$$

and

$$\int_M (|\nabla u_p|^2 + a u_p^2) dv_g = \int_M u_p^p dv_g + \omega^2 \int_M (1 - q\Phi(u_p))^2 u_p^2 dv_g. \quad (3.37)$$

Let  $(p_\alpha)_\alpha$  be a sequence such that  $p_\alpha < 6$  for all  $\alpha$  and  $p_\alpha \rightarrow 6$  as  $\alpha \rightarrow +\infty$ . Let  $v_\alpha = u_{p_\alpha}$ . Subtracting (3.36) -  $\frac{1}{2}$ (3.37), we get from (3.4) and (3.35) that the sequence consisting of the  $\|v_\alpha\|_{L^{p_\alpha}}$ 's is bounded. Then, by (3.36),  $(v_\alpha)_\alpha$  is bounded in  $H^1$ . In particular, there exists  $u \in H^1$ ,  $u \geq 0$ , such that  $v_\alpha \rightharpoonup u$  weakly in  $H^1$ ,  $v_\alpha \rightarrow u$  strongly in  $L^3$ , and  $v_\alpha \rightarrow u$  a.e. as  $\alpha \rightarrow +\infty$ . Since  $I'_{p_\alpha}(v_\alpha) = 0$ , it follows from (3.8)–(3.9) that there also holds that  $I'_6(u) = 0$ . By the Trudinger [72] regularity argument developed for critical Schrödinger equations, by standard elliptic regularity, and by the maximum principle, either  $u \equiv 0$ , or  $u > 0$  in  $M$  and  $u, \Phi(u)$  are smooth. Thus it remains only to prove that  $u \not\equiv 0$ . By the sharp Sobolev inequality, as established in Hebey and Vaugon [43, 44], there exists  $B > 0$  such that

$$\left( \int_M v_\alpha^{p_\alpha} dv_g \right)^{2/p_\alpha} \leq (K_3^2 + o(1)) \int_M (|\nabla v_\alpha|^2 + a v_\alpha^2) dv_g + B \int_M v_\alpha^2 dv_g \quad (3.38)$$

for all  $\alpha$ . By contradiction we assume  $u \equiv 0$ . Subtracting (3.36) -  $\frac{1}{p_\alpha}$ (3.37) we get that

$$\int_M (|\nabla v_\alpha|^2 + a v_\alpha^2) dv_g = \frac{2p_\alpha}{p_\alpha - 2} c_{p_\alpha} + o(1), \quad (3.39)$$

and we also have that

$$\int_M (|\nabla v_\alpha|^2 + a v_\alpha^2) dv_g = \int_M v_\alpha^{p_\alpha} dv_g + o(1). \quad (3.40)$$

Inserting (3.39)–(3.40) into (3.38) we obtain that

$$\left( \frac{2p_\alpha}{p_\alpha - 2} c_{p_\alpha} + o(1) \right)^{2/p_\alpha} \leq K_3^2 \left( \frac{2p_\alpha}{p_\alpha - 2} c_{p_\alpha} + o(1) \right) \quad (3.41)$$

for all  $\alpha$ , and the contradiction follows from (3.35) by letting  $\alpha \rightarrow +\infty$  in (3.41). This ends the proof of the existence part of Theorem 0.1 when we assume (0.6) with the property that the inequality in (0.6) is strict at some point when  $(M, g)$  is conformally diffeomorphic to the unit 3-sphere.  $\square$

At this point it remains to prove the existence part in Theorem 0.1 when  $p = 6$  and  $(M, g)$  is conformally diffeomorphic to the 3-sphere. This is the subject of what follows.

*Proof of the Existence Part in Theorem 0.1 when  $p = 6$ . Case 2.* We assume that  $(M, g)$  is conformally diffeomorphic to the unit 3-sphere. Without loss of generality we can assume that  $M = S^3$  and that  $g = \varphi^4 g_0$  for some smooth positive function

$\varphi > 0$ . We let  $(\beta_\alpha)_\alpha$  be any sequence of real numbers such that  $\beta_\alpha > 1$  for all  $\alpha$  and  $\beta_\alpha \rightarrow 1$  as  $\alpha \rightarrow +\infty$ . We fix  $x_0 \in S^3$  and define the functions  $\varphi_\alpha : S^3 \rightarrow \mathbb{R}$  by

$$\varphi_\alpha(x) = \frac{(3(\beta_\alpha^2 - 1))^{1/4}}{\varphi \sqrt{2(\beta_\alpha - \cos r)}}, \quad (3.42)$$

where  $r = d_{g_0}(x_0, x)$ . We have that

$$\Delta_g \varphi_\alpha + \frac{1}{8} S_g \varphi_\alpha = \varphi_\alpha^5 \quad (3.43)$$

and that

$$\left( \int_{S^3} \varphi_\alpha^6 dv_g \right)^{1/3} = K_3^2 \int_{S^3} \left( |\nabla \varphi_\alpha|^2 + \frac{1}{8} S_g \varphi_\alpha^2 \right) dv_g \quad (3.44)$$

for all  $\alpha$ . A possible reference in book form for (3.43) and (3.44) is Hebey [41]. It follows from (3.42)-(3.44) that

$$\int_{S^3} |\nabla \varphi_\alpha|^2 dv_g = \frac{1}{K_3^3} + o(1) \quad \text{and} \quad \int_{S^3} \varphi_\alpha^6 dv_g = \frac{1}{K_3^3} \quad (3.45)$$

for all  $\alpha$ , while  $\varphi_\alpha \rightarrow 0$  in  $L^q$  for all  $q < 6$  as  $\alpha \rightarrow +\infty$ . If  $\lambda = 0$  or  $\omega = 0$ , then, by (0.6), either  $a \equiv \frac{1}{8} S_g$  or  $a \leq \frac{1}{8} S_g$ , the inequality being strict at least at one point. In the first case, by (3.43), for any  $\alpha$ ,  $(\varphi_\alpha, \frac{1}{q})$  is a solution of (0.2) if  $\lambda = 0$  while  $(\varphi_\alpha, \Phi(\varphi_\alpha))$  is a solution of (0.2) if  $\lambda > 0$ . In the second case, by (3.44),

$$\frac{\int_{S^3} (|\nabla \varphi_\alpha|^2 + a \varphi_\alpha^2) dv_g}{\left( \int_{S^3} \varphi_\alpha^6 dv_g \right)^{1/3}} < \frac{1}{K_3^2}$$

and it follows from Aubin [3, 4] that there exists  $u$  smooth and positive such that

$$\Delta_g u + a u = u^5.$$

Then  $(u, \frac{1}{q})$  solves (0.2) if  $\lambda = 0$  while  $(u, \Phi(u))$  solves (0.2) if  $\lambda > 0$ . We may thus assume that  $\lambda > 0$  and  $\omega^2 > 0$ . By (3.45) there exists  $T > 0$ , independent of  $\alpha$ , such that  $I_6(T\varphi_\alpha) < 0$  for all  $\alpha \gg 1$ . Let  $(t_\alpha)_\alpha$  be any sequence in  $[0, T]$ . By (3.8), since  $\Phi(0) = 0$ , there holds that  $\Phi(t_\alpha \varphi_\alpha) \rightarrow 0$  in  $H^1$  as  $\alpha \rightarrow +\infty$ . By (3.2) and (3.4) we then get that  $\Phi(t_\alpha \varphi_\alpha) \rightarrow 0$  in  $H^{2,q}$  for all  $q < 3$  as  $\alpha \rightarrow +\infty$ . In particular,  $\Phi(t_\alpha \varphi_\alpha) \rightarrow 0$  in  $L^\infty$  as  $\alpha \rightarrow +\infty$ , and we can write that

$$\max_{0 \leq t \leq T} \|\Phi(t\varphi_\alpha)\|_{L^\infty} \rightarrow 0 \quad (3.46)$$

as  $\alpha \rightarrow +\infty$ . Since  $k\lambda < 1$ , it follows from (3.46) that there exists  $\varepsilon_0 > 0$  such that for any  $\alpha \gg 1$ , and any  $t \in [0, T]$ ,

$$\int_M (1 - q\Phi(t\varphi_\alpha)) u_\alpha^2 dv_g \geq \left( k\lambda + \frac{\varepsilon_0}{\omega^2} \right) \int_M u_\alpha^2 dv_g. \quad (3.47)$$

Let  $\mathcal{H}_6$  be the functional defined on  $H^1$  by

$$\mathcal{H}_6(u) = \frac{1}{2} \int_{S^3} |\nabla u|^2 dv_g + \frac{1}{2} \int_{S^3} \left( \frac{1}{8} S_g - \varepsilon_0 \right) u^2 dv_g - \frac{1}{6} \int_{S^3} |u|^6 dv_g. \quad (3.48)$$

Then, by (0.6) and (3.47),

$$\max_{0 \leq t \leq T} I_6(t\varphi_\alpha) \leq \max_{0 \leq t \leq T} \mathcal{H}_6(t\varphi_\alpha) \quad (3.49)$$

for all  $\alpha \gg 1$ . Fix  $u_0 = \varphi_\alpha$  for  $\alpha \gg 1$ , and let  $T_0 = T$ . Then, for  $\varepsilon > 0$  sufficiently small, we can write that  $I_p(T_0 u_0) < 0$  and

$$\max_{0 \leq t \leq T_0} I_p(tu_0) \leq (1 + \delta_\varepsilon) \max_{0 \leq t \leq T_0} I_6(tu_0) \quad (3.50)$$

for all  $p \in (6 - \varepsilon, 6)$ , where  $\delta_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . We have that

$$\max_{0 \leq t \leq T_0} \mathcal{H}_6(tu_0) \leq \left( \frac{1}{2} - \frac{1}{6} \right) \left( \frac{\int_{S^3} (|\nabla u_0|^2 + (\frac{1}{8}S_g - \varepsilon_0)u_0^2) dv_g}{(\int_{S^3} u_0^6 dv_g)^{1/3}} \right)^{3/2}. \quad (3.51)$$

By (3.43) and (3.44), since  $\varepsilon_0 > 0$ , it follows from (3.51) that

$$\max_{0 \leq t \leq T_0} \mathcal{H}_6(tu_0) < \frac{1}{3K_3^3}. \quad (3.52)$$

As a consequence of (3.16), (3.49), (3.50) and (3.52), we get that there exists  $\delta_0 > 0$  such that for  $0 < \varepsilon \ll 1$  sufficiently small,

$$\delta_0 \leq c_p \leq \frac{1}{3K_3^3} - \delta_0 \quad (3.53)$$

for all  $p \in (6 - \varepsilon, 6)$ . Then we conclude as in case 1 of the proof. This ends the proof of the existence part in Theorem 0.1.  $\square$

#### 4. PROOF OF THE UNIFORM BOUND IN THEOREM 0.1 WHEN $p \in (2, 6)$

We prove the uniform bounds in Theorem 0.1 when  $p \in (2, 6)$ . Let  $(\omega_\alpha)_\alpha$  be a sequence in  $(-\omega_a, \omega_a)$  such that  $\omega_\alpha \rightarrow \omega$  as  $\alpha \rightarrow +\infty$  for some  $\omega \in [-\omega_a, \omega_a]$ . Also let  $p \in (2, 6)$  and  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (0.2) with phases  $\omega_\alpha$ . Then,

$$\begin{cases} \Delta_g u_\alpha + a u_\alpha = u_\alpha^{p-1} + \omega_\alpha^2 (q v_\alpha - 1)^2 u_\alpha \\ \Delta_g v_\alpha + (\lambda + q^2 u_\alpha^2) v_\alpha = q u_\alpha^2 \end{cases} \quad (4.1)$$

for all  $\alpha$ . Since  $u_\alpha > 0$ , we get with the second equation in (4.1), see (3.5), that  $0 \leq v_\alpha \leq \frac{1}{q}$  for all  $\alpha$ . Assume by contradiction that

$$\max_M u_\alpha \rightarrow +\infty \quad (4.2)$$

as  $\alpha \rightarrow +\infty$ . Let  $x_\alpha \in M$  and  $\mu_\alpha > 0$  be given by

$$u_\alpha(x_\alpha) = \max_M u_\alpha = \mu_\alpha^{-2/(p-2)}.$$

By (4.2),  $\mu_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Define  $\tilde{u}_\alpha$  by

$$\tilde{u}_\alpha(x) = \mu_\alpha^{\frac{2}{p-2}} u_\alpha(\exp_{x_\alpha}(\mu_\alpha x))$$

and  $g_\alpha$  by  $g_\alpha(x) = (\exp_{x_\alpha}^* g)(\mu_\alpha x)$  for  $x \in B_0(\delta \mu_\alpha^{-1})$ , where  $\delta > 0$  is small. Since  $\mu_\alpha \rightarrow 0$ , we get that  $g_\alpha \rightarrow \xi$  in  $C_{loc}^2(\mathbb{R}^3)$  as  $\alpha \rightarrow +\infty$ . Moreover, by (4.1),

$$\Delta_{g_\alpha} \tilde{u}_\alpha + \mu_\alpha^2 \hat{a}_\alpha \tilde{u}_\alpha = \tilde{u}_\alpha^{p-1} + \omega_\alpha^2 \mu_\alpha^2 (q \hat{v}_\alpha - 1)^2 \tilde{u}_\alpha, \quad (4.3)$$

where  $\hat{a}_\alpha$  and  $\hat{v}_\alpha$  are given by

$$\hat{a}_\alpha(x) = a(\exp_{x_\alpha}(\mu_\alpha x)) \quad \text{and} \quad \hat{v}_\alpha(x) = v_\alpha(\exp_{x_\alpha}(\mu_\alpha x)).$$



In addition,  $\tilde{u}_\alpha(0) = 1$  and  $0 \leq \tilde{u}_\alpha \leq 1$ . By (4.3) and standard elliptic theory arguments, we can write that, after passing to a subsequence,  $\tilde{u}_\alpha \rightarrow u$  in  $C_{loc}^{1,\theta}(\mathbb{R}^3)$  as  $\alpha \rightarrow +\infty$ , where  $u$  is such that  $u(0) = 1$  and  $0 \leq u \leq 1$ . Then

$$\Delta_\xi u = u^{p-1}$$

in  $\mathbb{R}^3$ , where  $\Delta_\xi$  is the Euclidean Laplacian. It follows that  $u$  is actually smooth and positive, and, since  $2 < p < 6$ , we get a contradiction with the Liouville result of Gidas and Spruck [40]. As a conclusion, (4.2) is not possible and there exists  $C > 0$  such that

$$u_\alpha + v_\alpha \leq C \quad (4.4)$$

in  $M$  for all  $\alpha$ . Coming back to (4.1) it follows that the sequences  $(u_\alpha)_\alpha$  and  $(v_\alpha)_\alpha$  are actually bounded in  $H^{2,q}$  for all  $q$ . Pushing one step further the regularity argument they turn out to be bounded in  $H^{3,q}$  for all  $q$ , and by the Sobolev embedding theorem we get that they are also bounded in  $C^{2,\theta}$ ,  $0 < \theta < 1$ . This ends the proof of the uniform bounds in Theorem 0.1 when  $p \in (2, 6)$ .

As a remark on the above proof it is necessary to assume that  $u_\alpha \not\equiv 0$  since if not the case, when  $\lambda = 0$ , couples like  $(0, t)$  solve (0.2) for all  $t > 0$ . Assuming  $u_\alpha \geq 0$ ,  $u_\alpha \not\equiv 0$ , we get that  $u_\alpha > 0$  in  $M$  and also that  $v_\alpha > 0$  in  $M$ . An operator like  $\Delta_g + h$  is said to be coercive if its energy is a possible norm for  $H^1$  or, in an equivalent way, if there exists  $C > 0$  such that

$$C \int_M u^2 dv_g \leq \int_M (|\nabla u|^2 + hu^2) dv_g$$

for all  $u \in H^1$ . A complementary lemma is as follows.

**Lemma 4.1.** *Let  $(\omega_\alpha)_\alpha$  be a sequence in  $(-\omega_a, \omega_a)$  such that  $\omega_\alpha \rightarrow \omega$  as  $\alpha \rightarrow +\infty$  for some  $\omega \in [-\omega_a, \omega_a]$ ,  $p \in (2, 6]$ , and  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (4.1). Assume that the operator  $\Delta_g + (a - \omega^2)$  is coercive. Let  $u_\alpha \rightarrow u$  and  $v_\alpha \rightarrow v$  in  $C^2$  as  $\alpha \rightarrow +\infty$ . Then  $u > 0$ ,  $v > 0$ , and  $u, v$  are smooth solutions of (0.2).*

*Proof.* Assume that  $\Delta_g + (a - \omega^2)$  is coercive. Then, for  $\varepsilon > 0$  sufficiently small,  $\Delta_g + (a - \omega^2 - \varepsilon)$  is still coercive. Since  $u_\alpha > 0$  in  $M$  there holds that  $0 \leq v_\alpha \leq \frac{1}{q}$  for all  $\alpha$ . In particular, by (4.1) and the Sobolev inequality, for any  $\alpha \gg 1$  sufficiently large,

$$\begin{aligned} & \int_M (|\nabla u_\alpha|^2 + (a - \omega^2 - \varepsilon) u_\alpha^2) dv_g \\ & \leq \int_M |\nabla u_\alpha|^2 dv_g + \int_M a u_\alpha^2 dv_g - \omega_\alpha^2 \int_M (q v_\alpha^2 - 1)^2 u_\alpha^2 dv_g \\ & = \int_M u_\alpha^p dv_g \\ & \leq C \left( \int_M (|\nabla u_\alpha|^2 + (a - \omega^2 - \varepsilon) u_\alpha^2) dv_g \right)^{p/2} \end{aligned}$$

for some  $C > 0$  independent of  $\alpha$ . This implies  $u > 0$  and then  $v > 0$ . The lemma follows.  $\square$

As a remark, Lemma 4.1 does not hold anymore if we allow  $\Delta_g + (a - \omega^2)$  not to be coercive. Suppose  $\lambda > 0$ ,  $a > 0$  is a positive constant, and let  $(\varepsilon_\alpha)_\alpha$  be a

sequence of positive real numbers such that  $\varepsilon_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Assuming  $a$  is constant we have that  $\omega_a^2 = a$ . Let  $u_\alpha = \varepsilon_\alpha$  and

$$v_\alpha = \frac{q\varepsilon_\alpha^2}{\lambda + q^2\varepsilon_\alpha^2}.$$

Then  $u_\alpha \rightarrow 0$  and  $v_\alpha \rightarrow 0$  in  $C^2$  as  $\alpha \rightarrow +\infty$ , and we do have that  $(u_\alpha, v_\alpha)$  solves (4.1), where

$$\omega_\alpha^2 = \frac{1}{(qv_\alpha - 1)^2} (\omega_a^2 - \varepsilon_\alpha^{p-2}).$$

Noting that  $\omega_\alpha \rightarrow \omega_a$  as  $\alpha \rightarrow +\infty$ , the construction provides a counter example to Lemma 4.1 when  $\omega^2 = a$ . As an independent remark,  $\Delta_g + (a - \omega^2)$  is automatically coercive when  $\omega \in (-\omega_a, \omega_a)$ . Also we may allow  $\omega = \pm\omega_a$  if  $\Delta_g + (a - \min_M a)$  is coercive, and this is automatically the case if  $a$  is nonconstant.

## 5. SHARP BLOW-UP ESTIMATES WHEN $p = 6$

In what follows we let  $(M, g)$  be a smooth compact 3-dimensional Riemannian manifold,  $a > 0$  be a smooth positive function in  $M$ , and  $(\omega_\alpha)_\alpha$  be a sequence in  $(-\omega_a, \omega_a)$  such that  $\omega_\alpha \rightarrow \omega$  as  $\alpha \rightarrow +\infty$  for some  $\omega \in [-\omega_a, \omega_a]$ , where  $\omega_a$  is as in (0.1). Also we let  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (0.2) with phases  $\omega_\alpha$  and  $p = 6$ . Namely,

$$\begin{cases} \Delta_g u_\alpha + a u_\alpha = u_\alpha^5 + \omega_\alpha^2 (qv_\alpha - 1)^2 u_\alpha \\ \Delta_g v_\alpha + (\lambda + q^2 u_\alpha^2) v_\alpha = q u_\alpha^2 \end{cases} \quad (5.1)$$

for all  $\alpha$ . Since  $u_\alpha > 0$ , we get with the second equation in (5.1), see (3.5), that  $0 \leq v_\alpha \leq \frac{1}{q}$  for all  $\alpha$ . In particular, if we let

$$h_\alpha = a - \omega_\alpha^2 (qv_\alpha - 1)^2, \quad (5.2)$$

then  $\|h_\alpha\|_{L^\infty} \leq C$  for all  $\alpha$ , where  $C > 0$  is independent of  $\alpha$ . Assume by contradiction that

$$\max_M u_\alpha \rightarrow +\infty \quad (5.3)$$

as  $\alpha \rightarrow +\infty$ . In what follows we let  $(x_\alpha)_\alpha$  be a sequence of points in  $M$ , and  $(\rho_\alpha)_\alpha$  be a sequence of positive real numbers,  $0 < \rho_\alpha < i_g/7$  for all  $\alpha$ , where  $i_g$  is the injectivity radius of  $(M, g)$ . We assume that the  $x_\alpha$ 's and  $\rho_\alpha$ 's satisfy

$$\begin{cases} \nabla u_\alpha(x_\alpha) = 0 \text{ for all } \alpha, \\ d_g(x_\alpha, x)^{\frac{1}{2}} u_\alpha(x) \leq C \text{ for all } x \in B_{x_\alpha}(7\rho_\alpha) \text{ and all } \alpha, \\ \lim_{\alpha \rightarrow +\infty} \rho_\alpha^{\frac{1}{2}} \sup_{B_{x_\alpha}(6\rho_\alpha)} u_\alpha(x) = +\infty. \end{cases} \quad (5.4)$$

We let  $\mu_\alpha$  be given by

$$\mu_\alpha = u_\alpha(x_\alpha)^{-2}. \quad (5.5)$$

Since the  $h_\alpha$ 's in (5.2) are  $L^\infty$ -bounded we can apply the asymptotic analysis in Druet and Hebey [34] and Druet, Hebey and V  tois [37]. Closely related arguments were first developed by Schoen [65], and then by Druet [31] and Li and Zhu [53] assuming  $C^1$ -convergences of the potentials. Assuming (5.4), and coming back to the analysis in Druet, Hebey and V  tois [37], we can write that  $\frac{\rho_\alpha}{\mu_\alpha} \rightarrow +\infty$  as  $\alpha \rightarrow +\infty$  and that

$$\mu_\alpha^{\frac{1}{2}} u_\alpha(\exp_{x_\alpha}(\mu_\alpha x)) \rightarrow \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}} \quad (5.6)$$

in  $C_{loc}^1(\mathbb{R}^3)$  as  $\alpha \rightarrow +\infty$ , where  $\mu_\alpha$  is as in (5.5). In particular we have that  $\mu_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Now we define  $\varphi_\alpha : (0, \rho_\alpha) \mapsto \mathbb{R}^+$  by

$$\varphi_\alpha(r) = \frac{1}{|\partial B_{x_\alpha}(r)|_g} \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g, \quad (5.7)$$

where  $|\partial B_{x_\alpha}(r)|_g$  is the volume of the sphere of center  $x_\alpha$  and radius  $r$  for the induced metric. Let  $\Lambda = 2\sqrt{3}$ . We define  $r_\alpha \in [\Lambda\mu_\alpha, \rho_\alpha]$  by

$$r_\alpha = \sup \left\{ r \in [\Lambda\mu_\alpha, \rho_\alpha] \text{ s.t. } \left( s^{\frac{1}{2}} \varphi_\alpha(s) \right)' \leq 0 \text{ in } [\Lambda\mu_\alpha, r] \right\}. \quad (5.8)$$

It follows from (5.6) that

$$\frac{r_\alpha}{\mu_\alpha} \rightarrow +\infty \quad (5.9)$$

as  $\alpha \rightarrow +\infty$ , while the definition of  $r_\alpha$  gives that

$$r^{\frac{1}{2}} \varphi_\alpha \text{ is non-increasing in } [\Lambda\mu_\alpha, r_\alpha] \quad (5.10)$$

and that

$$\left( r^{\frac{1}{2}} \varphi_\alpha(r) \right)'(r_\alpha) = 0 \text{ if } r_\alpha < \rho_\alpha. \quad (5.11)$$

We prove that the following sharp asymptotic estimates on the  $u_\alpha$ 's in (5.1) hold true.

**Lemma 5.1.** *Let  $(M, g)$  be a smooth compact Riemannian 3-dimensional manifold, and  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (5.1) such that (5.3) holds true. Let  $(x_\alpha)_\alpha$  and  $(\rho_\alpha)_\alpha$  be such that (5.4) hold true, and let  $R \geq 6$  be such that  $Rr_\alpha \leq 6\rho_\alpha$  for all  $\alpha \gg 1$ . There exists  $C > 0$  such that, after passing to a subsequence,*

$$u_\alpha(x) + d_g(x_\alpha, x) |\nabla u_\alpha(x)| \leq C \mu_\alpha^{\frac{1}{2}} d_g(x_\alpha, x)^{-1} \quad (5.12)$$

for all  $x \in B_{x_\alpha}(\frac{R}{2}r_\alpha) \setminus \{x_\alpha\}$  and all  $\alpha$ , where  $\mu_\alpha$  is as in (5.5), and where  $r_\alpha$  is as in (5.8).

*Proof of Lemma 5.1.* Given  $R > 0$  we define

$$\eta_{R,\alpha} = \sup_{B_{x_\alpha}(Rr_\alpha) \setminus B_{x_\alpha}(\frac{1}{R}r_\alpha)} u_\alpha. \quad (5.13)$$

We prove that there exist  $C, C' > 0$  such that

$$u_\alpha(x) \leq C \left( \mu_\alpha^{\frac{1}{2}} d_g(x_\alpha, x)^{-1} + \eta_{R,\alpha} \right) \quad (5.14)$$

for all  $x \in B_{x_\alpha}(\frac{R}{2}r_\alpha) \setminus \{x_\alpha\}$  and all  $\alpha$ , and such that

$$\eta_{R,\alpha} \leq C' \mu_\alpha^{\frac{1}{2}} r_\alpha^{-1} \quad (5.15)$$

for all  $\alpha$ . Let  $R' \geq 6$  be given. The Harnack inequality in Druet, Hebey and Vétois [37] can be stated in the following way: there exists  $C > 1$  such that for any sequence  $(s_\alpha)_\alpha$  of positive real numbers satisfying that  $s_\alpha > 0$  and  $R's_\alpha \leq 6\rho_\alpha$  for all  $\alpha$ , there holds

$$s_\alpha \|\nabla u_\alpha\|_{L^\infty(\Omega_\alpha)} \leq C \sup_{\Omega_\alpha} u_\alpha \leq C^2 \inf_{\Omega_\alpha} u_\alpha, \quad (5.16)$$

where  $\Omega_\alpha = B_{x_\alpha}(R's_\alpha) \setminus B_{x_\alpha}(\frac{1}{R'}s_\alpha)$ . Lemma 5.1 follows from (5.14), (5.15), and (5.16) in order to get the gradient part in (5.12). We start with the proof of (5.14).

For this aim we let  $(y_\alpha)_\alpha$  be an arbitrary sequence in  $B_{x_\alpha}(\frac{R}{2}r_\alpha) \setminus \{x_\alpha\}$ , and prove that there exists  $C > 0$  such that, up to a subsequence,

$$u_\alpha(y_\alpha) \leq C \left( \mu_\alpha^{\frac{1}{2}} d_g(x_\alpha, y_\alpha)^{-1} + \eta_{R,\alpha} \right). \quad (5.17)$$

As a preliminary remark one can note that (5.17) directly follows from (5.6) if  $d_g(x_\alpha, y_\alpha) = O(\mu_\alpha)$ . By (5.16) we may then assume that

$$\lim_{\alpha \rightarrow +\infty} \frac{1}{\mu_\alpha} d_g(x_\alpha, y_\alpha) = +\infty \quad \text{and} \quad \lim_{\alpha \rightarrow +\infty} \frac{1}{r_\alpha} d_g(x_\alpha, y_\alpha) = 0. \quad (5.18)$$

Without loss of generality, since the  $\|h_\alpha\|_{L^\infty}$ 's are bounded, we can assume that, up to a subsequence,  $\|h_\alpha\|_{L^\infty} \rightarrow \Lambda$  as  $\alpha \rightarrow +\infty$  for some  $\Lambda \geq 0$ , where the  $h_\alpha$ 's are as in (5.2). Now we let  $k > 1$  be such that  $k\Lambda \notin \text{Sp}(\Delta_g)$ , where  $\text{Sp}(\Delta_g)$  is the spectrum of  $\Delta_g$ , and let  $G$  be the Green's function of  $\Delta_g - k\Lambda$ . Then, see, for instance, Robert [61], there are positive constants  $C_1 > 1$  and  $C_2, C_3 > 0$  such that

$$\begin{aligned} \frac{1}{C_1} d_g(x, y)^{-1} - C_2 &\leq G(x, y) \leq C_1 d_g(x, y)^{-1}, \quad \text{and} \\ |\nabla G(x, y)| &\leq C_3 d_g(x, y)^{-2} \end{aligned} \quad (5.19)$$

for all  $x \neq y$ . By (5.19) there exists  $\delta > 0$  such that  $G \geq 0$  in  $B_{x_\alpha}(\delta r_\alpha)$  for all  $\alpha$ . By (5.18),  $y_\alpha \in B_{x_\alpha}(\frac{\delta}{2}r_\alpha)$  for  $\alpha \gg 1$ , and by the Green's representation formula,

$$\begin{aligned} u_\alpha(y_\alpha) &= \int_{B_{x_\alpha}(\delta r_\alpha)} G(y_\alpha, x) (\Delta_g u_\alpha - k\Lambda u_\alpha)(x) dv_g(x) \\ &\quad + \int_{\partial B_{x_\alpha}(\delta r_\alpha)} G(y_\alpha, x) (\partial_\nu u_\alpha)(x) d\sigma_g(x) \\ &\quad - \int_{\partial B_{x_\alpha}(\delta r_\alpha)} (\partial_\nu G(y_\alpha, x)) u_\alpha(x) d\sigma_g(x), \end{aligned} \quad (5.20)$$

where  $\nu$  is the unit outward normal to  $\partial B_{x_\alpha}(\delta r_\alpha)$ . Since  $k > 1$ , and  $\|h_\alpha\|_{L^\infty} \rightarrow \Lambda$  as  $\alpha \rightarrow +\infty$ ,

$$\Delta_g u_\alpha - k\Lambda u_\alpha \leq u_\alpha^5$$

and since  $G \geq 0$  in  $B_{x_\alpha}(\delta r_\alpha)$  we get with (5.19) that

$$\begin{aligned} &\int_{B_{x_\alpha}(\delta r_\alpha)} G(y_\alpha, x) (\Delta_g u_\alpha - k\Lambda u_\alpha)(x) dv_g(x) \\ &\leq C \int_{B_{x_\alpha}(\delta r_\alpha)} d_g(y_\alpha, x)^{-1} u_\alpha(x)^5 dv_g(x). \end{aligned} \quad (5.21)$$

Also, by (5.16) and (5.19), we have that

$$\begin{aligned} &\int_{\partial B_{x_\alpha}(\delta r_\alpha)} G(y_\alpha, x) |\partial_\nu u_\alpha(x)| d\sigma_g(x) \leq C \eta_{R,\alpha}, \quad \text{and} \\ &\int_{\partial B_{x_\alpha}(\delta r_\alpha)} |\partial_\nu G(y_\alpha, x)| u_\alpha(x) d\sigma_g(x) \leq C \eta_{R,\alpha} \end{aligned} \quad (5.22)$$

for some  $C > 0$ . Combining (5.20)–(5.22), we get that

$$u_\alpha(y_\alpha) \leq C \int_{B_{x_\alpha}(\delta r_\alpha)} d_g(y_\alpha, x)^{-1} u_\alpha^5(x) dv_g(x) + C \eta_{R,\alpha}. \quad (5.23)$$

Following Druet, Hebey and Vétois [37], there holds that

$$u_\alpha(x) \leq C \left( \mu_\alpha^{1/10} d_g(x_\alpha, x)^{-3/5} + \eta_{R,\alpha} r_\alpha^{2/5} d_g(x_\alpha, x)^{-2/5} \right) \quad (5.24)$$

for all  $x \in B_{x_\alpha}(Rr_\alpha) \setminus \{x_\alpha\}$  and all  $\alpha$ , where  $C > 0$  does not depend on  $x$  and  $\alpha$ . In particular, we get with (5.4), (5.6), (5.18), and (5.24), that

$$\int_{B_{x_\alpha}(\delta r_\alpha)} d_g(y_\alpha, x)^{-1} u_\alpha^5(x) dv_g(x) = O \left( \mu_\alpha^{\frac{1}{2}} d_g(x_\alpha, y_\alpha)^{-1} \right) + O(\eta_{R,\alpha}) . \quad (5.25)$$

By (5.23) and (5.25), we obtain (5.17). In particular, (5.14) holds true. Now it remains to prove (5.15). By (5.10), for any  $\eta \in (0, 1)$ ,

$$(\eta r_\alpha)^{\frac{1}{2}} \varphi_\alpha(\eta r_\alpha) \geq r_\alpha^{\frac{1}{2}} \varphi_\alpha(r_\alpha)$$

for all  $\alpha \gg 1$ , where  $\varphi_\alpha$  is as in (5.7). By (5.16), there exists  $C > 1$  such that

$$\frac{1}{C} \sup_{B_{x_\alpha}(Rs_\alpha) \setminus B_{x_\alpha}(\frac{1}{R}s_\alpha)} u_\alpha \leq \varphi_\alpha(s_\alpha) \leq C \inf_{B_{x_\alpha}(Rs_\alpha) \setminus B_{x_\alpha}(\frac{1}{R}s_\alpha)} u_\alpha \quad (5.26)$$

for all  $0 < s_\alpha \leq r_\alpha$  and all  $\alpha$ . By (5.26) we then get that

$$\frac{1}{C} r_\alpha^{\frac{1}{2}} \eta_{R,\alpha} \leq (\eta r_\alpha)^{\frac{1}{2}} \sup_{\partial B_{x_\alpha}(\eta r_\alpha)} u_\alpha .$$

Assuming (5.14) it follows that

$$\frac{1}{C} \eta_{R,\alpha} \leq \eta^{\frac{1}{2}} \left( \mu_\alpha^{\frac{1}{2}} (\eta r_\alpha)^{-1} + \eta_{R,\alpha} \right)$$

and if we choose  $\eta \in (0, 1)$  sufficiently small such that  $C\eta^{\frac{1}{2}} \leq \frac{1}{2}$ , we obtain that

$$\eta_{R,\alpha} \leq \eta^{2-n} \mu_\alpha^{\frac{1}{2}} r_\alpha^{-1} .$$

In particular, (5.15) holds true. This ends the proof of the lemma.  $\square$

Now that we have Lemma 5.1 we prove that the following fundamental asymptotic estimate holds true.

**Lemma 5.2.** *Let  $(M, g)$  be a smooth compact Riemannian 3-dimensional manifold and  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (5.1) such that (5.3) holds true. Let  $(x_\alpha)_\alpha$  and  $(\rho_\alpha)_\alpha$  be such that (5.4) holds true. Assume  $r_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ , where  $r_\alpha$  is as in (5.8). Then  $\rho_\alpha = O(r_\alpha)$  and*

$$r_\alpha \mu_\alpha^{-\frac{1}{2}} u_\alpha(\exp_{x_\alpha}(r_\alpha x)) \rightarrow \frac{\sqrt{3}}{|x|} + \mathcal{H}(x) \quad (5.27)$$

in  $C_{loc}^2(B_0(2) \setminus \{0\})$  as  $\alpha \rightarrow +\infty$ , where  $\mu_\alpha$  is as in (5.5), and  $\mathcal{H}$  is a harmonic function in  $B_0(2)$  which satisfies that  $\mathcal{H}(0) = 0$ .

*Proof of Lemma 5.2.* Let  $R \geq 6$  be such that  $Rr_\alpha \leq 6\rho_\alpha$  for  $\alpha \gg 1$ . In what follows we assume that  $r_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . For  $x \in B_0(3)$  we set

$$\begin{aligned} \tilde{u}_\alpha(x) &= r_\alpha \mu_\alpha^{-\frac{1}{2}} u_\alpha(\exp_{x_\alpha}(r_\alpha x)) , \\ g_\alpha(x) &= (\exp_{x_\alpha}^* g)(r_\alpha x) , \text{ and} \\ \tilde{h}_\alpha(x) &= h_\alpha(\exp_{x_\alpha}(r_\alpha x)) , \end{aligned}$$

where  $h_\alpha$  is as in (5.2). Since  $r_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ , we have that  $\tilde{g}_\alpha \rightarrow \xi$  in  $C_{loc}^2(\mathbb{R}^3)$  as  $\alpha \rightarrow +\infty$ , where  $\xi$  is the Euclidean metric. Thanks to Lemma 5.1 we also have that

$$|\tilde{u}_\alpha(x)| \leq C|x|^{-1} \quad (5.28)$$

in  $B_0(\frac{R}{2}) \setminus \{0\}$ . By (5.1), (5.9), and thanks to standard elliptic theory we can write that, after passing to a subsequence,  $\tilde{u}_\alpha \rightarrow \tilde{u}$  in  $C_{loc}^2(B_0(\frac{R}{2}) \setminus \{0\})$  as  $\alpha \rightarrow +\infty$ , where  $\tilde{u}$  satisfies  $\Delta \tilde{u} = 0$  in  $B_0(\frac{R}{2}) \setminus \{0\}$ . By (5.28),  $|\tilde{u}(x)| \leq C|x|^{-1}$  in  $B_0(\frac{R}{2}) \setminus \{0\}$ . Thus we can write that

$$\tilde{u}(x) = \frac{\Lambda}{|x|} + \mathcal{H}(x) \quad (5.29)$$

where  $\Lambda \geq 0$  and  $\mathcal{H}$  satisfies  $\Delta \mathcal{H} = 0$  in  $B_0(\frac{R}{2})$ . In order to see that  $\Lambda = \sqrt{3}$ , it is sufficient to integrate the equation satisfied by the  $\tilde{u}_\alpha$ 's in  $B_0(1)$ . Then

$$-\int_{\partial B_0(1)} \partial_\nu \tilde{u}_\alpha d\sigma_{g_\alpha} = \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \int_{B_0(1)} \tilde{u}_\alpha^5 dv_{g_\alpha} - r_\alpha^2 \int_{B_0(1)} \tilde{h}_\alpha \tilde{u}_\alpha dv_{g_\alpha}, \quad (5.30)$$

where  $\nu$  is the unit outward normal derivative to  $\partial B_0(1)$ . By (5.28), the  $\tilde{u}_\alpha$ 's are bounded in  $L^1(B_0(1))$ . Changing  $x$  into  $\frac{\mu_\alpha}{r_\alpha}x$ , thanks to (5.6) and Lemma 5.1, we also have that

$$\lim_{\alpha \rightarrow +\infty} \left(\frac{\mu_\alpha}{r_\alpha}\right)^2 \int_{B_0(1)} \tilde{u}_\alpha^5 dv_{g_\alpha} = \int_{\mathbb{R}^3} \left(\frac{1}{1 + \frac{|x|^2}{3}}\right)^{5/2} dx = \sqrt{3}\omega_2. \quad (5.31)$$

Noting that by (5.29),

$$\lim_{\alpha \rightarrow +\infty} \int_{\partial B_0(1)} \partial_\nu \tilde{u}_\alpha d\sigma_{g_\alpha} = -\omega_2 \Lambda, \quad (5.32)$$

we get that  $\Lambda = \sqrt{3}$  by combining (5.30)–(5.32). Now we prove that  $\mathcal{H}(0) = 0$ . In what follows we let  $X_\alpha$  be the 1-form given by

$$X_\alpha(x) = \left(1 - \frac{1}{12} Rc_g^\sharp(x) (\nabla f_\alpha(x), \nabla f_\alpha(x))\right) \nabla f_\alpha(x), \quad (5.33)$$

where  $f_\alpha(x) = \frac{1}{2}d_g(x_\alpha, x)^2$  and, in local coordinates,  $(Rc_g^\sharp)^{ij} = g^{i\mu}g^{j\nu}R_{\mu\nu}$ , where the  $R_{ij}$ 's are the components of the Ricci curvature  $Rc_g$  of  $g$ . We adopt the notations that  $A^\sharp$  is the musical isomorphism applied to  $A$ , and that  $X(\nabla u) = (X, \nabla u)$  for  $X$  a 1-form and  $u$  a function. By the Pohozaev identity in Druet and Hebey [35], that we apply to the  $u_\alpha$ 's in  $B_{x_\alpha}(r_\alpha)$  with the above choice of  $X_\alpha$ , we have that

$$\begin{aligned} & \int_{B_{x_\alpha}(r_\alpha)} X_\alpha(\nabla u_\alpha) h_\alpha u_\alpha dv_g + \frac{1}{12} \int_{B_{x_\alpha}(r_\alpha)} (\Delta_g \operatorname{div}_g X_\alpha) u_\alpha^2 dv_g \\ & + \frac{1}{6} \int_{B_{x_\alpha}(r_\alpha)} (\operatorname{div}_g X_\alpha) h_\alpha u_\alpha^2 dv_g = Q_{1,\alpha} + Q_{2,\alpha} + Q_{3,\alpha}, \end{aligned} \quad (5.34)$$

where

$$\begin{aligned} Q_{1,\alpha} &= \frac{1}{6} \int_{\partial B_{x_\alpha}(r_\alpha)} (\operatorname{div}_g X_\alpha) (\partial_\nu u_\alpha) u_\alpha d\sigma_g \\ &\quad - \int_{\partial B_{x_\alpha}(r_\alpha)} \left( \frac{1}{2} X_\alpha(\nu) |\nabla u_\alpha|^2 - X_\alpha(\nabla u_\alpha) \partial_\nu u_\alpha \right) d\sigma_g, \end{aligned}$$

$$Q_{2,\alpha} = - \int_{B_{x_\alpha}(r_\alpha)} \left( \nabla X_\alpha - \frac{1}{3} (\operatorname{div}_g X_\alpha) g \right)^\sharp (\nabla u_\alpha, \nabla u_\alpha) dv_g ,$$

$$Q_{3,\alpha} = \frac{1}{6} \int_{\partial B_{x_\alpha}(r_\alpha)} X_\alpha(\nu) u_\alpha^6 d\sigma_g - \frac{1}{12} \int_{\partial B_{x_\alpha}(r_\alpha)} (\partial_\nu (\operatorname{div}_g X_\alpha)) u_\alpha^2 d\sigma_g ,$$

and  $\nu$  is the unit outward normal derivative to  $\partial B_{x_\alpha}(r_\alpha)$ . We have that

$$(\nabla X_\alpha)_{ij} - \frac{1}{n} (\operatorname{div}_g X_\alpha) g_{ij} = O(d_g(x_\alpha, x)^2) \quad (5.35)$$

for all  $i, j$ . By Lemma 5.1 and (5.35) we then get that

$$\begin{aligned} |Q_{2,\alpha}| &\leq C \int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha, x)^2 |\nabla u_\alpha(x)|^2 dv_g(x) \\ &\leq C \mu_\alpha \int_{B_{x_\alpha}(r_\alpha)} d_g(x_\alpha, x)^{-2} dv_g(x) \\ &\leq C \mu_\alpha r_\alpha . \end{aligned} \quad (5.36)$$

Similarly,

$$\begin{aligned} |X_\alpha(x)| &= O(d_g(x_\alpha, x)) , \\ \operatorname{div}_g X_\alpha(x) &= 3 + O(d_g(x_\alpha, x)^2) , \text{ and} \\ \Delta_g (\operatorname{div}_g X_\alpha)(x) &= \frac{3}{2} S_g(x_\alpha) + O(d_g(x_\alpha, x)) . \end{aligned} \quad (5.37)$$

By (5.37) and Lemma 5.1, we then get that

$$\begin{aligned} &\int_{B_{x_\alpha}(r_\alpha)} X_\alpha(\nabla u_\alpha) h_\alpha u_\alpha dv_g + \frac{1}{12} \int_{B_{x_\alpha}(r_\alpha)} (\Delta_g \operatorname{div}_g X_\alpha) u_\alpha^2 dv_g \\ &+ \frac{1}{6} \int_{B_{x_\alpha}(r_\alpha)} (\operatorname{div}_g X_\alpha) h_\alpha u_\alpha^2 dv_g = O(\mu_\alpha r_\alpha) , \end{aligned} \quad (5.38)$$

and since there also holds that  $|\nabla (\operatorname{div}_g X_\alpha)(x)| = O(d_g(x_\alpha, x))$ , we can write in addition that

$$|Q_{3,\alpha}| = O(\mu_\alpha^3 r_\alpha^{-3}) + O(\mu_\alpha r_\alpha) . \quad (5.39)$$

Combining (5.34), (5.36), (5.38), and (5.39) we get that

$$\begin{aligned} &\frac{1}{6} \int_{\partial B_{x_\alpha}(r_\alpha)} (\operatorname{div}_g X_\alpha) (\partial_\nu u_\alpha) u_\alpha d\sigma_g \\ &- \int_{\partial B_{x_\alpha}(r_\alpha)} \left( \frac{1}{2} X_\alpha(\nu) |\nabla u_\alpha|^2 - X_\alpha(\nabla u_\alpha) \partial_\nu u_\alpha \right) d\sigma_g \\ &= O(\mu_\alpha^3 r_\alpha^{-3}) + O(\mu_\alpha r_\alpha) . \end{aligned} \quad (5.40)$$

Since  $\tilde{u}_\alpha \rightarrow \tilde{u}$  in  $C_{loc}^2(B_0(\frac{R}{2}) \setminus \{0\})$  as  $\alpha \rightarrow +\infty$ , where  $\tilde{u}$  satisfies  $\Delta \tilde{u} = 0$  and  $|\tilde{u}(x)| \leq C|x|^{-1}$  in  $B_0(\frac{R}{2}) \setminus \{0\}$ , and by (5.29), we independently get that

$$\begin{aligned} &\frac{1}{6} \int_{\partial B_{x_\alpha}(r_\alpha)} (\operatorname{div}_g X_\alpha) (\partial_\nu u_\alpha) u_\alpha d\sigma_g \\ &- \int_{\partial B_{x_\alpha}(r_\alpha)} \left( \frac{1}{2} X_\alpha(\nu) |\nabla u_\alpha|^2 - X_\alpha(\nabla u_\alpha) \partial_\nu u_\alpha \right) d\sigma_g \\ &= \left( \frac{3}{2} \omega_2 \Lambda \mathcal{H}(0) + o(1) \right) \frac{\mu_\alpha}{r_\alpha} . \end{aligned} \quad (5.41)$$

Combining (5.40) and (5.41), we get with (5.9) that  $\mathcal{H}(0) = 0$ . At this point it remains to prove that  $\rho_\alpha = O(r_\alpha)$ . We proceed by contradiction and assume that  $r_\alpha \rho_\alpha^{-1} \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Then (5.29) holds in  $B_0(R) \setminus \{0\}$  for all  $R$ , and  $r_\alpha < \rho_\alpha$  for  $\alpha \gg 1$ . In particular, we get with (5.11) and (5.29) that  $(r^{1/2}\varphi(r))'(1) = 0$ , where

$$\begin{aligned} \varphi(r) &= \frac{1}{\omega_2 r^2} \int_{\partial B_0(r)} \tilde{u} d\sigma \\ &= \frac{\Lambda}{r^{n-2}} + \mathcal{H}(0) . \end{aligned}$$

Since  $\mathcal{H}(0) = 0$ , it follows that  $\Lambda = 0$ , and this is impossible since  $\Lambda = \sqrt{3}$ . Lemma 5.2 is proved.  $\square$

## 6. PROOF OF THE UNIFORM BOUND IN THEOREM 0.1 WHEN $p = 6$

We prove that the uniform bound in the theorem holds true when  $p = 6$ . For this aim we use the analysis developed in Section 5 to prove that blow-up points are isolated. Then we use phase compensation, and the positive mass theorem of Schoen, and Yau [68] (see also Schoen and Yau [69, 70] Witten [74]) to prove that there are no blow-up points when we assume (0.6) with the property that (0.6) is strict at least at one point if  $(M, g)$  is conformally diffeomorphic to the unit 3-sphere and  $\omega\lambda = 0$ .

Here again we let  $(M, g)$  be a smooth compact 3-dimensional Riemannian manifold,  $a > 0$  be a smooth positive function in  $M$ , and  $(\omega_\alpha)_\alpha$  be a sequence in  $(-\omega_a, \omega_a)$  such that  $\omega_\alpha \rightarrow \omega$  as  $\alpha \rightarrow +\infty$  for some  $\omega \in [-\omega_a, \omega_a]$ , where  $\omega_a$  is as in (0.1). Also we let  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (0.2) with phases  $\omega_\alpha$  and  $p = 6$ . In particular, the  $u_\alpha$ 's and  $v_\alpha$ 's satisfy (5.1). We assume that (5.3) holds true. Following Druet and Hebey [34], see also Druet, Hebey and Vétois [37], there exists  $C > 0$  such that for any  $\alpha$  the following holds true. Namely that there exist  $N_\alpha \in \mathbb{N}^*$  and  $N_\alpha$  critical points of  $u_\alpha$ , denoted by  $(x_{1,\alpha}, x_{2,\alpha}, \dots, x_{N_\alpha,\alpha})$ , such that

$$d_g(x_{i,\alpha}, x_{j,\alpha})^{\frac{1}{2}} u_\alpha(x_{i,\alpha}) \geq 1 \quad (6.1)$$

for all  $i, j \in \{1, \dots, N_\alpha\}$ ,  $i \neq j$ , and

$$\left( \min_{i=1, \dots, N_\alpha} d_g(x_{i,\alpha}, x) \right)^{\frac{1}{2}} u_\alpha(x) \leq C \quad (6.2)$$

for all  $x \in M$  and all  $\alpha$ . We define  $d_\alpha$  by

$$d_\alpha = \min_{1 \leq i < j \leq N_\alpha} d_g(x_{i,\alpha}, x_{j,\alpha}) . \quad (6.3)$$

If  $N_\alpha = 1$ , we set  $d_\alpha = \frac{1}{4}i_g$ , where  $i_g$  is the injectivity radius of  $(M, g)$ . The first important lemma we prove in this section is that blow-up points are necessarily isolated in the sense that  $d_\alpha \not\rightarrow 0$  as  $\alpha \rightarrow +\infty$ .

**Lemma 6.1.** *Let  $(M, g)$  be a smooth compact Riemannian 3-dimensional manifold and  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (5.1) such that (5.3) holds true. Then  $d_\alpha \not\rightarrow 0$  as  $\alpha \rightarrow +\infty$ , where  $d_\alpha$  is as in (6.3).*



*Proof of Lemma 6.1.* We proceed by contradiction and assume that  $d_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Then  $N_\alpha \geq 2$  for  $\alpha \gg 1$ , and we can assume that the  $x_{i,\alpha}$ 's are such that  $d_g(x_{1,\alpha}, x_{i,\alpha}) \leq d_g(x_{1,\alpha}, x_{i+1,\alpha})$  for all  $i = 2, \dots, N_\alpha$ . Let  $\delta \in (0, \frac{1}{2}i_g)$  be given. For  $x \in B_0(\delta d_\alpha^{-1})$ , we let

$$\tilde{u}_\alpha(x) = d_\alpha^{1/2} u_\alpha \left( \exp_{x_{1,\alpha}}(d_\alpha x) \right). \quad (6.4)$$

We let also  $\tilde{h}_\alpha(x) = h_\alpha \left( \exp_{x_{1,\alpha}}(d_\alpha x) \right)$ , and  $\tilde{g}_\alpha(x) = \left( \exp_{x_{1,\alpha}}^* g \right) (d_\alpha x)$ , where  $h_\alpha$  is as in (5.2). Then, by (5.1),

$$\Delta_{\tilde{g}_\alpha} \tilde{u}_\alpha + d_\alpha^2 \tilde{h}_\alpha \tilde{u}_\alpha = \tilde{u}_\alpha^5, \quad (6.5)$$

and we clearly have that  $\tilde{g}_\alpha \rightarrow \xi$  in  $C_{loc}^2(\mathbb{R}^3)$  as  $\alpha \rightarrow +\infty$ . Given  $R > 0$  we let  $1 \leq N_{R,\alpha} \leq N_\alpha$  be such that  $d_g(x_{1,\alpha}, x_{i,\alpha}) \leq R d_\alpha$  for all  $1 \leq i \leq N_{R,\alpha}$ , and  $d_g(x_{1,\alpha}, x_{i,\alpha}) > R d_\alpha$  for all  $N_{R,\alpha} + 1 \leq i \leq N_\alpha$ . We have that  $N_{R,\alpha} \geq 2$  for all  $R > 1$ , and  $(N_{R,\alpha})_\alpha$  is uniformly bounded for all  $R > 0$ . Mimicking the arguments in Druet and Hebey [34], see also Druet, Hebey and Vétois [37], given  $R > 0$ , there holds that

$$\begin{aligned} & \text{either } \tilde{u}_\alpha(\tilde{x}_{i,\alpha}) = O(1) \text{ for all } 1 \leq i \leq N_{R,\alpha}, \\ & \text{or } \tilde{u}_\alpha(\tilde{x}_{i,\alpha}) \rightarrow +\infty \text{ as } \alpha \rightarrow +\infty \text{ for all } 1 \leq i \leq N_{R,\alpha}, \end{aligned} \quad (6.6)$$

where the  $\tilde{u}_\alpha$ 's are as in (6.4), and

$$\tilde{x}_{i,\alpha} = \frac{1}{d_\alpha} \exp_{x_{1,\alpha}}^{-1}(x_{i,\alpha}). \quad (6.7)$$

Now we split the proof into the study of two cases. In the first case we assume that there exist  $R > 0$  and  $1 \leq i \leq N_{R,\alpha}$  such that  $\tilde{u}_\alpha(\tilde{x}_{i,\alpha}) = O(1)$ . Then, by (6.6),  $\tilde{u}_\alpha(\tilde{x}_{i,\alpha}) = O(1)$  for all  $1 \leq i \leq N_{R,\alpha}$  and all  $R > 0$ . Noting that the two first equations in (5.4) are satisfied by  $x_\alpha = x_{i,\alpha}$  and  $\rho_\alpha = \frac{1}{8}d_\alpha$ , it follows from (5.6) that the sequence  $(\tilde{u}_\alpha)_\alpha$  is uniformly bounded in the balls  $B_{\tilde{x}_{i,\alpha}}(1/2)$ . Thus, by (6.5) and elliptic theory, the sequence  $(\tilde{u}_\alpha)_\alpha$  is bounded in  $C_{loc}^1(\mathbb{R}^3)$ . Up to a subsequence, still thanks to (6.5), we get that the  $\tilde{u}_\alpha$ 's converge in  $C_{loc}^1(\mathbb{R}^3)$  as  $\alpha \rightarrow +\infty$  to some  $\tilde{u}$  which satisfies  $\Delta \tilde{u} = \tilde{u}^5$  in  $\mathbb{R}^3$ . Moreover,  $\tilde{u}$  has two critical points which are 0 and the limit  $\tilde{x}_2 \in S^2$  as  $\alpha \rightarrow +\infty$  of the  $\tilde{x}_{2,\alpha}$ 's in (6.7). By the classification result of Caffarelli, Gidas, and Spruck [26], this is impossible. In particular, we are left with the second case of our study, where we assume that there exist  $R > 0$  and  $1 \leq i \leq N_{R,\alpha}$  such that  $\tilde{u}_\alpha(\tilde{x}_{i,\alpha}) \rightarrow +\infty$  as  $\alpha \rightarrow +\infty$ . Then, by (6.6),  $\tilde{u}_\alpha(\tilde{x}_{i,\alpha}) \rightarrow +\infty$  as  $\alpha \rightarrow +\infty$  for all  $1 \leq i \leq N_{R,\alpha}$  and all  $R > 0$ . The assumptions (5.4) are satisfied by  $x_\alpha = x_{1,\alpha}$  and  $\rho_\alpha = \frac{1}{8}d_\alpha$ . Let  $\tilde{v}_\alpha = \tilde{u}_\alpha(0)\tilde{u}_\alpha$ . By (6.5),

$$\Delta_{\tilde{g}_\alpha} \tilde{v}_\alpha + d_\alpha^2 \tilde{h}_\alpha \tilde{v}_\alpha = \frac{1}{\tilde{u}_\alpha(0)^4} \tilde{v}_\alpha^5. \quad (6.8)$$

Noting that  $\tilde{u}_\alpha(0) \rightarrow +\infty$  as  $\alpha \rightarrow +\infty$ , mimicking again arguments from Druet and Hebey [34], and Druet, Hebey and Vétois [37], we get with (6.8) that, up to a subsequence,  $\tilde{u}_\alpha(0)\tilde{u}_\alpha \rightarrow \tilde{G}$  in  $C_{loc}^1(\mathbb{R}^3 \setminus \{\tilde{x}_i\}_{i \in I})$  as  $\alpha \rightarrow +\infty$ , where the  $\tilde{x}_i$ 's are the limits of the  $\tilde{x}_{i,\alpha}$ 's in (6.7), and  $I = \{1, \dots, \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} N_{R,\alpha}\}$ . Moreover,

for any  $R > 0$ ,

$$\begin{aligned} \tilde{G}(x) &= \sum_{i=1}^{\tilde{N}_R} \frac{\Lambda_i}{|x - \tilde{x}_i|} + \tilde{H}_R(x) \\ &= \frac{\Lambda_1}{|x|} + \left( \sum_{i=2}^{\tilde{N}_R} \frac{\Lambda_i}{|x - \tilde{x}_i|} + \tilde{H}_R(x) \right) \end{aligned} \quad (6.9)$$

in  $B_0(R)$ , where  $\Lambda_i > 0$  for all  $i$ ,  $\tilde{H}_R$  is harmonic in  $B_0(R)$ ,  $2 \leq \tilde{N}_R \leq N_{2R}$  is such that  $|\hat{x}_{\tilde{N}_R}| \leq R$  and  $|\hat{x}_{\tilde{N}_R+1}| > R$ , and  $N_{2R,\alpha} \rightarrow N_{2R}$  as  $\alpha \rightarrow +\infty$ . By Lemma 5.2, and (6.9), we get that  $\Lambda_1 = \sqrt{3}$  and that

$$\sum_{i=2}^{\tilde{N}_R} \frac{\Lambda_i}{|\tilde{x}_i|} + \tilde{H}_R(0) = 0. \quad (6.10)$$

Independently, by the maximum principle, since  $\tilde{G} \geq 0$  and  $|\tilde{x}_2| = 1$ , there holds that

$$\sum_{i=2}^{\tilde{N}_R} \frac{\Lambda_i}{|\tilde{x}_i|} + \tilde{H}_R(0) \geq \Lambda_2 - \frac{\sqrt{3}}{R} - \frac{\Lambda_2}{(R-1)}. \quad (6.11)$$

Choosing  $R \gg 1$  sufficiently large, we get a contradiction by combining (6.10) and (6.11). In particular,  $d_\alpha \not\rightarrow 0$  as  $\alpha \rightarrow +\infty$ , and this proves Lemma 6.1.  $\square$

Now that we know that blow-up points are isolated, we use elliptic theory and phase compensation to get strong convergence of the potential term in the nonlinear equation in (5.1). Namely we prove that the following lemma holds true.

**Lemma 6.2.** *Let  $(M, g)$  be a smooth compact Riemannian 3-dimensional manifold and  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (5.1) such that (5.3) holds true. Then  $(u_\alpha)_\alpha$  is bounded in  $H^1$  and, up to a subsequence,  $u_\alpha \rightarrow 0$  in  $H^1$  and  $v_\alpha \rightarrow v$  in  $C^{0,\theta}$  as  $\alpha \rightarrow +\infty$ , where  $v$  is a constant and  $0 < \theta < 1$ . Moreover, if  $\lambda > 0$ , then  $v = 0$  and  $h_\alpha \rightarrow a - \omega^2$  in  $C^{0,\theta}$  as  $\alpha \rightarrow +\infty$ , where  $0 < \theta < 1$ ,  $\omega$  is the limit of the  $\omega_\alpha$ 's, and  $h_\alpha$  is as in (5.2).*

*Proof of Lemma 6.2.* By Lemma 6.1, the sequence  $(N_\alpha)_\alpha$  is uniformly bounded. Up to a subsequence we may assume that  $N_\alpha = N$  for all  $\alpha$ . Without loss of generality we may also assume that for any  $\delta > 0$ ,

$$\sup_{B_{x_{i,\alpha}}(\delta)} u_\alpha \rightarrow +\infty$$

as  $\alpha \rightarrow +\infty$  for all  $i = 1, \dots, N$ . It follows that there exists  $\delta_0 > 0$ , sufficiently small, such that (5.4) holds true with  $x_\alpha = x_{i,\alpha}$  and  $\rho_\alpha = \delta_0$  for all  $i = 1, \dots, N$ . We fix  $i = 1, \dots, N$  arbitrary. By Lemma 5.2,  $r_\alpha \not\rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Then it follows from Lemma 5.1 that there exist  $r > 0$  and  $C > 0$  such that

$$u_\alpha(x) \leq C \mu_\alpha^{\frac{1}{2}} d_g(x_\alpha, x)^{-1} \quad (6.12)$$

for all  $x \in B_{x_\alpha}(r) \setminus \{x_\alpha\}$  and all  $\alpha$ , where  $\mu_\alpha$  is as in (5.5). In particular, together with (5.6), this implies that

$$\begin{aligned} \int_{B_{x_\alpha}(r)} u_\alpha^6 dv_g &= \int_{B_{x_\alpha}(\mu_\alpha)} u_\alpha^6 dv_g + \int_{B_{x_\alpha}(r) \setminus B_{x_\alpha}(\mu_\alpha)} u_\alpha^6 dv_g \\ &\leq C \end{aligned}$$

for all  $\alpha$ , and since the  $h_\alpha$ 's in (5.2) are bounded in  $L^\infty$ , we get with (5.1) that the  $u_\alpha$ 's are actually bounded in  $H^1$ . As a first consequence, since

$$\Delta_g v_\alpha + (\lambda + q^2 u_\alpha^2) v_\alpha = q u_\alpha^2, \quad (6.13)$$

and  $0 \leq v_\alpha \leq \frac{1}{q}$ , we get that the  $v_\alpha$ 's are such that the  $(\Delta_g v_\alpha + v_\alpha)$ 's are bounded in  $L^3$ . By elliptic theory it follows the  $v_\alpha$ 's are bounded in  $H^{2,3}$ , and we can write that, up to a subsequence,

$$v_\alpha \rightarrow v \quad (6.14)$$

in  $C^{0,\theta}$  as  $\alpha \rightarrow +\infty$  for some  $v$ , where  $0 < \theta < 1$ . Up to a subsequence, since the  $u_\alpha$ 's are bounded in  $H^1$ , we can assume that  $u_\alpha \rightarrow u$  in  $H^1$  as  $\alpha \rightarrow +\infty$  for some  $u \in H^1$ . Let  $x_i$  be the limit of the  $x_{i,\alpha}$ 's as  $\alpha \rightarrow +\infty$ ,  $i = 1, \dots, N$ . By (5.4), the  $u_\alpha$ 's are bounded in  $L_{loc}^\infty(M \setminus S)$ , where  $S = \{x_1, \dots, x_N\}$ . By elliptic estimates it follows that  $u_\alpha \rightarrow u$  in  $C_{loc}^1(M \setminus S)$  as  $\alpha \rightarrow +\infty$ . Coming back to (6.12) we then get that, necessarily,  $u \equiv 0$  in  $\bigcup_{i=1}^N B_{x_i}(r)$ . By (6.14), we can assume that  $h_\alpha \rightarrow h$  in  $C^{0,\theta}$  as  $\alpha \rightarrow +\infty$  for some  $h$ , and  $u$  solves  $\Delta_g u + hu = u^5$ . In particular, the maximum principle applies and we actually have that  $u \equiv 0$  in  $M$ . By Rellich-Kondrakov we can assume that  $u_\alpha \rightarrow 0$  in  $L^p$  as  $\alpha \rightarrow +\infty$  for  $p < 6$ . Then, by (6.13),  $\Delta_g v + \lambda v = 0$  in  $M$ . In particular,  $v$  is a constant, and if  $\lambda > 0$ , then  $v = 0$ . This proves Lemma 6.2.  $\square$

In what follows we let  $\delta > 0$  be given, sufficiently small, and let  $\eta \in C^\infty(M \times M)$ ,  $0 \leq \eta \leq 1$ , be such that  $\eta(x, y) = 1$  if  $d_g(x, y) \leq \delta$  and  $\eta(x, y) = 0$  if  $d_g(x, y) \geq 2\delta$ . For  $x \neq y$  we define

$$H(x, y) = \frac{\eta(x, y)}{\omega_2 d_g(x, y)}, \quad (6.15)$$

where  $\omega_2$  is the volume of the unit 2-sphere. The following lemma, which will be used in the proof of the uniform bound in Theorem 0.1 when  $p = 6$ , establishes basic estimates for the Green's functions of Schrödinger's operators as well as a positive mass property for such operators that we deduce from the maximum principle and the positive mass theorem of Schoen and Yau [68].

**Lemma 6.3.** *Let  $(M, g)$  be a smooth compact Riemannian 3-dimensional manifold and  $\Lambda \in C^\infty(M)$  be such that  $\Delta_g + \Lambda$  is coercive. The Green's function  $G$  of  $\Delta_g + \Lambda$  can be written as*

$$G(x, y) = H(x, y) + R(x, y) \quad (6.16)$$

for all  $(x, y) \in M \times M \setminus D$ , where  $D$  is the diagonal in  $M \times M$ , and  $R$  is continuous in  $M \times M$ . Moreover, for any  $x \in M$ , there exists  $C > 0$  such that

$$d_g(x, y) |\nabla R_x(y)| \leq C \quad (6.17)$$

for all  $y \in M \setminus \{x\}$ , where  $R_x(y) = R(x, y)$ , and there also holds that

$$\delta_\alpha \max_{y \in \partial B_x(\delta_\alpha)} |\nabla R_x(y)| = o(1) \quad (6.18)$$

for all sequences  $(\delta_\alpha)_\alpha$  of positive real numbers converging to zero. At last, if we assume that  $\Lambda \leq \frac{1}{8} S_g$ , the inequality being strict at least at one point if  $(M, g)$  is conformally diffeomorphic to the unit 3-sphere, then  $R(x, x) > 0$  for all  $x \in M$ .

*Proof of Lemma 6.3.* The decomposition (6.16) is well known. It is also known that there exists  $C > 0$  such that  $d_g(x, y)|\Delta_g R_x(y)| \leq C$  for all  $y \in M \setminus \{x\}$ . Possible references for such properties are Aubin [3], or Druet, Hebey and Robert [36]. We refer also to Robert [61]. Now we establish (6.17) and (6.18). We let  $(y_\alpha)_\alpha$  be an arbitrary sequence in  $M \setminus \{x\}$  such that  $y_\alpha \rightarrow x$  as  $\alpha \rightarrow +\infty$ . Let  $\delta_\alpha = d_g(x, y_\alpha)$  and  $R_\alpha(y) = R_x(\exp_x(\delta_\alpha y))$  for  $y \in \mathbb{R}^3$ . Let also  $g_\alpha$  be the metric given by  $g_\alpha(y) = (\exp_x^* g)(\delta_\alpha y)$ , and  $\tilde{y}_\alpha \in \mathbb{R}^3$  be such that  $y_\alpha = \exp_x(\delta_\alpha \tilde{y}_\alpha)$ . There holds that  $|\Delta_g R_\alpha(y)| \leq C\delta_\alpha|y|^{-1}$ , that  $(R_\alpha)_\alpha$  is bounded in  $L^\infty$ , and that  $g_\alpha \rightarrow \xi$  in  $C_{loc}^1(\mathbb{R}^3)$  as  $\alpha \rightarrow +\infty$ , where  $\xi$  is the Euclidean metric. There also holds that  $|\tilde{y}_\alpha| = 1$  for all  $\alpha$ . Let  $\tilde{y}$  be such that  $\tilde{y}_\alpha \rightarrow \tilde{y}$  as  $\alpha \rightarrow +\infty$ . Since  $|\tilde{y}| = 1$  it follows from standard elliptic theory that  $(R_\alpha)_\alpha$  is bounded in the  $C^1$ -topology in the Euclidean ball of center  $\tilde{y}$  and radius  $1/4$ . Since  $(y_\alpha)_\alpha$  is arbitrary, this proves (6.17). Noting that  $\Delta_{g_\alpha} R_\alpha \rightarrow 0$  uniformly in compact subsets of  $\mathbb{R}^3 \setminus \{0\}$  as  $\alpha \rightarrow +\infty$ , we get that  $R_\alpha \rightarrow R$  in  $C_{loc}^1(\mathbb{R}^3)$  as  $\alpha \rightarrow +\infty$ , where  $R$  is harmonic and bounded in  $\mathbb{R}^3 \setminus \{0\}$ . By Liouville's theorem,  $R$  is constant. This implies (6.18). Now it remains to prove the positive mass property that  $R(x, x) > 0$  for all  $x$  if we assume that  $\Lambda \leq \frac{1}{8}S_g$ , the inequality being strict at least at one point if  $(M, g)$  is conformally diffeomorphic to the unit 3-sphere. Let  $\tilde{G}$  be the Green's function of the conformal Laplacian  $\Delta_g + \frac{1}{8}S_g$ . Let  $x \in M$  and  $h \geq 0$  be smooth and such that

$$h \leq \left( \frac{1}{8}S_g - \Lambda \right) \tilde{G}_x.$$

If  $\Lambda \equiv \frac{1}{8}S_g$ , then  $h \equiv 0$ , but if not the case we can take  $h \not\equiv 0$ . Let  $\tilde{h}$ , smooth, be such that  $\Delta_g \tilde{h} + \Lambda \tilde{h} = h$ . Then  $\tilde{h} \geq 0$  and  $\tilde{h} \not\equiv 0$  if  $h \not\equiv 0$ . In particular, by the maximum principle,  $\tilde{h} > 0$  in  $M$  if  $\tilde{h} \not\equiv 0$ . Let  $\mathcal{H} = G_x - \tilde{G}_x - \tilde{h}$ . Noting that

$$\Delta_g \mathcal{H} + \Lambda \mathcal{H} \geq 0$$

in  $M$ , and that by the local expansions of  $G_x$  and  $\tilde{G}_x$ ,  $\mathcal{H}$  is continuous in  $M$ , we get from the maximum principle that  $\mathcal{H} \geq 0$  in  $M$ . In particular, by (3.24),  $R(x, x) \geq A + \tilde{h}(x)$ , and we get that  $R(x, x) > 0$  by the positive mass theorem of Schoen and Yau [68]. This ends the proof of the lemma.  $\square$

Thanks to Lemma 6.2 and Lemma 6.3 we can now prove the uniform bound in Theorem 0.1 when  $p = 6$ . In the process we use the asymptotic control we obtained in Lemma 5.1.

*Proof of the uniform bound in Theorem 0.1 when  $p = 6$ .* In what follows we consider a smooth compact 3-dimensional Riemannian manifold  $(M, g)$ , and let  $a > 0$  be a smooth positive function in  $M$ ,  $\omega \in (-\omega_a, \omega_a)$ , and  $(\omega_\alpha)_\alpha$  be a sequence such that  $\omega_\alpha \rightarrow \tilde{\omega}$  as  $\alpha \rightarrow +\infty$  for some  $\tilde{\omega} \in [-\omega_a, -\omega] \cup [\omega, \omega_a]$ , where  $\omega_a$  is as in (0.1). We assume either that  $|\tilde{\omega}| < \omega_a$ , or that  $\Delta_g + (a - \omega_a^2)$  is a coercive operator. In case  $a$  is constant and  $\tilde{\omega} = \omega_a$ , we apply the arguments in Section 7, noting that  $v_\alpha = 1/q$  for all  $\alpha$  in case  $\lambda = 0$ . We let  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of (0.2) with phases  $\omega_\alpha$  and  $p = 6$ . In particular, the  $u_\alpha$ 's and  $v_\alpha$ 's satisfy (5.1). We assume by contradiction that (5.3) holds true and we assume (0.6), with the property that (0.6) is strict at least at one point if  $(M, g)$  is conformally diffeomorphic to the unit 3-sphere and  $\omega\lambda = 0$ . By Lemma 6.1, the sequence  $(N_\alpha)_\alpha$  is uniformly bounded. Up to a subsequence we may assume that  $N_\alpha = N$

for all  $\alpha$ . We let  $x_i$  be the limit of the  $x_{i,\alpha}$ 's as  $\alpha \rightarrow +\infty$ , and let the  $\mu_{i,\alpha}$  be as in (5.5) given by

$$\mu_{i,\alpha} = u_\alpha(x_{i,\alpha})^{-2}$$

for all  $i = 1, \dots, N$  and all  $\alpha$ . Without loss of generality we can assume that  $\mu_{i,\alpha} \rightarrow 0$  as  $\alpha \rightarrow +\infty$  for all  $i$ . We reorganize the  $i$ 's such that, up to a subsequence,

$$\mu_{1,\alpha} = \max_i \mu_{i,\alpha}$$

and we define  $\mu_i \geq 0$  by

$$\mu_i = \lim_{\alpha \rightarrow +\infty} \frac{\mu_{i,\alpha}}{\mu_{1,\alpha}}. \quad (6.19)$$

By Lemma 5.1 and the Harnack inequality for any  $\delta > 0$  there exists  $C > 0$  such that

$$u_\alpha \leq C \mu_{1,\alpha}^{1/2} \quad (6.20)$$

in  $M \setminus \bigcup_{i=1}^N B_{x_{i,\alpha}}(\delta)$  for all  $\alpha$ . There holds that

$$\Delta_g(\mu_{1,\alpha}^{-1/2} u_\alpha) + h_\alpha(\mu_{1,\alpha}^{-1/2} u_\alpha) = \mu_{1,\alpha}^2 (\mu_{1,\alpha}^{-1/2} u_\alpha)^5 \quad (6.21)$$

for all  $\alpha$ , where  $h_\alpha$  is as in (5.2). By Lemma 6.2, the  $h_\alpha$ 's converge in  $C^{0,\theta}$ ,  $\theta \in (0, 1)$ . Let  $h$  be the limit of the  $h_\alpha$ 's. Still by Lemma 6.2,  $h = a - \tilde{\omega}^2$  if  $\lambda > 0$ . In general,  $h = a - \tilde{\omega}^2(qv - 1)^2$  so that  $h \geq a - \tilde{\omega}^2 \geq a - \omega_a^2$ . By our assumptions that either  $|\tilde{\omega}| < \omega_a$ , or  $\Delta_g + (a - \omega_a^2)$  is coercive, we get that  $\Delta_g + h$  is coercive. Combining (6.20) and (6.21), we get thanks to standard elliptic theory that

$$\mu_{1,\alpha}^{-1/2} u_\alpha \rightarrow \mathcal{H} \quad (6.22)$$

in  $C_{loc}^1(M \setminus S)$  as  $\alpha \rightarrow +\infty$ , where  $S = \{x_1, \dots, x_N\}$ . By (5.1), Lemma 5.1, (6.20) and the convergence of the  $h_\alpha$ 's to  $h$ , we can write that

$$\Delta_g \mathcal{H} + h \mathcal{H} = \sqrt{3} \omega_2 \sum_{i=1}^N \mu_i^{1/2} \delta_{x_i}. \quad (6.23)$$

Since  $\Delta_g + h$  is coercive, we get from (6.23) that

$$\mathcal{H}(x) = \sqrt{3} \omega_2 \sum_{i=1}^N \mu_i^{1/2} (H(x_i, x) + R(x_i, x)) \quad (6.24)$$

where  $H$  and  $R$  are as in Lemma 6.3 with  $\Lambda = h$ . Let  $i = 1, \dots, N$  be arbitrary and  $X_\alpha$  be the 1-form given by  $X_\alpha = \nabla f_\alpha$ , where  $f_\alpha(x) = \frac{1}{2} d_g(x_{i,\alpha}, x)^2$ . We apply the Pohozaev identity in Druet and Hebey [35] to  $u_\alpha$  in  $B_{x_{i,\alpha}}(r)$ ,  $r > 0$  small. By Lemma 5.1, multiplying the Pohozaev identity by  $\mu_{1,\alpha}^{-1}$ , letting  $\alpha \rightarrow +\infty$  tend to infinity, we get that

$$\begin{aligned} & \frac{1}{6} \int_{\partial B_{x_i}(r)} (\operatorname{div}_g X) \mathcal{H} \partial_\nu \mathcal{H} d\sigma_g \\ & - \int_{\partial B_{x_i}(r)} \left( \frac{1}{2} X(\nu) |\nabla \mathcal{H}|^2 - (X, \nabla \mathcal{H}) \partial_\nu \mathcal{H} \right) d\sigma_g \\ & = \frac{1}{12} \int_{\partial B_{x_i}(r)} (\partial_\nu (\operatorname{div}_g X)) \mathcal{H}^2 d\sigma_g + o(1), \end{aligned} \quad (6.25)$$

where  $X = \nabla f$ ,  $f = \frac{1}{2} d_g(x_i, \cdot)^2$ ,  $\nu$  is the unit outward normal derivative to  $\partial B_{x_i}(r)$ , and  $o(1) \rightarrow 0$  as  $r \rightarrow 0$ . We have that  $\operatorname{div}_g X = 3 + O(d_g(x_i, x)^2)$  and that

$|\nabla \operatorname{div}_g X| = O(d_g(x_i, x))$ . By Lemma 5.1 there also holds that  $|\mathcal{H}| \leq C d_g(x_i, \cdot)^{-1}$  in  $M \setminus \{x_i\}$ . Thanks to (6.17) we then get that

$$\lim_{r \rightarrow 0} \int_{\partial B_{x_i}(r)} (\partial_\nu (\operatorname{div}_g X)) \mathcal{H}^2 d\sigma_g = 0 \quad (6.26)$$

and that

$$\frac{1}{6} \int_{\partial B_{x_i}(r)} (\operatorname{div}_g X) \mathcal{H} \partial_\nu \mathcal{H} d\sigma_g = \frac{1}{2} \int_{\partial B_{x_i}(r)} \mathcal{H} \partial_\nu \mathcal{H} d\sigma_g + o(1) \quad (6.27)$$

as  $r \rightarrow 0$ . Choosing  $\delta > 0$  in the definition of  $\eta$  in (6.15) such that  $d_g(x_j, x_k) \geq 4\delta$  for all  $j, k = 1, \dots, N$  such that  $x_j \neq x_k$ , we get that  $\mathcal{R}(x_j, x_i) \geq 0$  for all  $j \neq i$ . By (6.18) and (6.24), we compute

$$\begin{aligned} & \frac{1}{2} \int_{\partial B_{x_i}(r)} \mathcal{H} \partial_\nu \mathcal{H} d\sigma_g - \int_{\partial B_{x_i}(r)} \left( \frac{1}{2} X(\nu) |\nabla \mathcal{H}|^2 - (X, \nabla \mathcal{H}) \partial_\nu \mathcal{H} \right) d\sigma_g \\ &= -\frac{3\omega_2}{2} \mu_i^{1/2} \sum_{j=1}^N \mu_j^{1/2} \mathcal{R}(x_j, x_i) + o(1). \end{aligned} \quad (6.28)$$

Combining (6.25)–(6.28), letting  $r \rightarrow 0$ , it follows that

$$\mu_i^{1/2} \mu_j^{1/2} \mathcal{R}(x_j, x_i) = 0 \quad (6.29)$$

for all  $i, j$ . Letting  $i, j = 1$ , we have that  $\mu_1 = 1$ , and it follows from (6.29) that  $\mathcal{R}(x_1, x_1) = 0$ . By assumption,  $h < \frac{1}{8} S_g$  if  $\omega\lambda \neq 0$ , and in case  $\omega\lambda = 0$ , we get that  $h \leq a \leq \frac{1}{8} S_g$  with the property that either the manifold is not conformally diffeomorphic to the unit 3-sphere, or that the inequality is strict at one point. By Lemma 6.3 it follows that  $\mathcal{R}(x_1, x_1) > 0$  and we get a contradiction. This proves that there exists  $C > 0$  such that  $\|u_\alpha\|_{L^\infty} \leq C$  for all  $\alpha$  and all  $\omega_\alpha \in K(\omega)$ . Since we also have that  $0 \leq v_\alpha \leq \frac{1}{q}$  for all  $\alpha$ , it follows from elliptic theory and (5.1) that the  $u_\alpha$ 's and  $v_\alpha$ 's are bounded in  $H^{2,p}$  for all  $p > 1$ . The  $C^{2,\theta}$ -bound easily follows. This ends the proof of Theorem 0.1.  $\square$

## 7. PROOF OF THEOREM 2.1

We use here part of the analysis developed in Sections 5 and 6, together with a nice concluding argument from Brézis and Li [25]. As in the proof of Theorem 0.1 we proceed by contradiction. We let  $((u_\alpha, v_\alpha))_\alpha$  be a sequence of smooth positive solutions of

$$\begin{cases} \Delta_g u_\alpha + a_\alpha u_\alpha = u_\alpha^5 + \omega_\alpha^2 (q v_\alpha - 1)^2 u_\alpha \\ \Delta_g v_\alpha + (\lambda + q^2 u_\alpha^2) v_\alpha = q u_\alpha^2 \end{cases} \quad (7.1)$$

such that  $\max_M u_\alpha \rightarrow +\infty$  as  $\alpha \rightarrow +\infty$ , where the  $a_\alpha$ 's are smooth positive functions and the  $\omega_\alpha$ 's are phases in  $(-\omega_{a_\alpha}, \omega_{a_\alpha})$  such that

- (i) either  $a_\alpha - \omega_\alpha^2 \rightarrow 0$  in  $L^\infty(M)$  as  $\alpha \rightarrow +\infty$  and  $\lambda > 0$ , or
- (ii)  $a_\alpha \rightarrow 0$  in  $L^\infty(M)$  and  $\omega_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ , and  $\lambda \geq 0$ .

The estimates in Section 5 as well as Lemma 6.1, which establishes that  $d_\alpha \not\rightarrow 0$  as  $\alpha \rightarrow +\infty$ , still hold true in the present situation. Let  $h_\alpha$  be given by

$$h_\alpha = a_\alpha - \omega_\alpha^2 (q v_\alpha - 1)^2.$$

Without loss of generality, up to passing to a subsequence, we may assume that  $a_\alpha \rightarrow a$  in  $C^{0,\theta}$  and that  $\omega_\alpha \rightarrow \omega$  as  $\alpha \rightarrow +\infty$ . Then, by the arguments developed in Lemma 6.2, we get that  $v_\alpha \rightarrow v$  in  $C^{0,\theta}$  as  $\alpha \rightarrow +\infty$ , and we then get with

(i) and (ii) that  $h_\alpha \rightarrow 0$  in  $C^{0,\theta}$  as  $\alpha \rightarrow +\infty$ . As in the proof of Theorem 0.1 in Section 6, the convergence  $\mu_{1,\alpha}^{-1/2} u_\alpha \rightarrow \mathcal{H}$  in (6.22) holds true. However, in the present situation,  $h \equiv 0$  and we get that  $\Delta_g \mathcal{H} = 0$  in  $M \setminus S$ , while

$$\Delta_g \mathcal{H} = \sqrt{3}\omega_2 \sum_{i=1}^N \mu_i^{1/2} \delta_{x_i}$$

in the sense of distributions, where we adopt the notations of Section 6. In other words,  $\mathcal{H}$  is a nonnegative harmonic function with poles, and this is impossible. Indeed, from now on, without loss of generality, we may assume that  $\mu_i > 0$  for all  $i$  (we know that at least  $\mu_1 > 0$ ). Then  $\mathcal{H} \geq 0$  and  $\mathcal{H}$  is not constant. Let  $G$  be a Green's function of  $\Delta_g$ , and  $G_i = G(x_i, \cdot)$ . By regularity theory,

$$\mathcal{H} = \sqrt{3}\omega_2 \sum_{i=1}^N \mu_i^{1/2} G_i + \mathcal{F},$$

where  $\mathcal{F}$  is smooth. In particular, by standard expansion of the Green's function at its pole, see, for instance, Aubin [4], there exists  $x \in M \setminus S$  where  $\mathcal{H}$  attains its minimum. By the maximum principle we would get that  $\mathcal{H}$  is actually constant in  $M \setminus S$ , a contradiction.

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